# Answers to Problem Set # 01, 18.306

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1 Boundary cooling of initially uniform temperature

1.1 Statement: Boundary cooling of initially uniform temperature

Consider the 1-D heat equation problem, with constant heat diffusion coefficient $\nu > 0$,

$$T_t = \nu T_{xx} \quad \text{for} \quad 0 < x, t < \infty,$$

with boundary condition $T(0, t) = 0$, $t > 0$, \hspace{1cm} (1.1)

and initial condition $T(x, 0) = T_0$, $x > 0$, where $T_0 > 0$ is a constant temperature.

Use dimensional analysis to reduce the pde problem to an ode problem, and find the solution (explicitly). Plot the solution for some (small) time $t > 0$.

1.2 Answer: Boundary cooling of initially uniform temperature

There is only one input parameter that has temperature dimensions in it. Hence the solution must have the form $T = T_0 u(x, t)$, where $u$ has no dimensions. In turn, $u$ must be a function only of nondimensional combinations of the variables and the parameters in the problem. The only\(^1\) such combination is $z = \frac{x}{2\sqrt{\nu t}}$. It follows that the solution must have the form

$$T = T_0 f(z), \quad z = \frac{x}{2\sqrt{\nu t}} > 0,$$

where $f$ is some function to be found. Furthermore, from the initial and boundary conditions we see that $f$ must satisfy

$$f(0) = 1 \quad \text{and} \quad \lim_{z \to \infty} f(z) = 1.$$ \hspace{1cm} (1.3)

Substituting the form in (1.2) into (1.1) yields

$$-2z \frac{df}{dz} = \frac{df}{dz^2} \iff \frac{df}{dz} = a e^{-z^2} \quad \text{for some constant} \ a.$$ \hspace{1cm} (1.4)

Thus, from (1.3), $f(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds$. That is, $f$ is the error function erf, so that the solution to (1.1) is (see figure 1.1)

$$T = T_0 \text{erf} \left( \frac{x}{2\sqrt{\nu t}} \right).$$ \hspace{1cm} (1.5)

2 Conservation of probability in QM

2.1 Conservation of probability in QM

In non-relativistic quantum mechanics the motion of a point particle in a potential $V$ is described by Schrödinger’s equation.

$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \psi_{xx} + V(x) \psi \quad \text{in} \ 1D,$$ \hspace{1cm} (2.1)

where $\hbar = \frac{\hbar}{2\pi}$ is the Plank constant divided by $2\pi$, $\psi = \psi(x, t)$ is the (complex valued) wave function, $m$ the particle’s mass, and $i$ is the imaginary unit.

The interpretation is that $^2$

$$\bar{\rho} = |\psi|^2 = \psi \psi^*$$ \hspace{1cm} (2.2)

\(^1\) Only in the sense that any other must be a function of $z$, e.g.: $(z + 1)^2$ or $\sin z$.

\(^2\) Here $^*$ indicates the complex conjugate.
Figure 1.1: Boundary cooling of initially uniform temperature. The solution in (1.5) is self-similar, and has the same shape for all $t > 0$. The figure indicates that there is a “boundary layer” — of approximate width $0 < x < 4 \sqrt{\nu t}$ — where the solution adjusts from the boundary condition, to the initial data (which fails to satisfy $T = 0$ at $x = 0$).

is the pdf [probability distribution function] (pdf) for the particle position. That is, the probability of finding the particle in any interval $a < x < b$ is

$$\int_a^b \tilde{\rho} \, dx.$$  \hfill (2.3)

Now: probability is conserved, and $\tilde{\rho}$ is its density. **Question: What is the probability flux?**

**Hint.** Use (2.1) to find an equation of the form $\tilde{\rho}_t + \tilde{q}_x = 0$. The flux is then $\tilde{q}$.

**Warning:** check that the flux you obtain is real valued.

### 2.2 Answer: Conservation of probability in QM

Multiply (2.1) by $\psi^*$. Take the resulting equation, and subtract from it its complex conjugate. After a bit of manipulation, the result can be written in the form

$$i \hbar \tilde{\rho}_t + \frac{\hbar^2}{2m} (\psi_x \psi^* - \psi \psi^*_x)_x = 0. \hfill (2.4)$$

Thus the probability flux is

$$\tilde{q} = \frac{\hbar}{2i m} (\psi_x \psi^* - \psi \psi^*_x), \hfill (2.5)$$

which is real valued, as expected.

Alternatively, write the wave function using polar variables $\psi = r e^{i \theta}$, where $\tilde{\rho} = r^2$. Substituting this into (2.1), multiplying by $e^{-i \theta}$, and taking real and imaginary parts, leads to the two equations

$$\hbar r_t = -\frac{\hbar^2}{2m} (2r_x \theta_x + r \theta_{xx}), \hfill (2.6)$$

$$\hbar r \theta_t = \frac{\hbar^2}{2m} (r_{xx} - r \theta_x^2) - V r. \hfill (2.7)$$

Multiplying (2.6) by $\frac{\hbar}{r}$ yields

$$\tilde{\rho}_t + \left( \frac{\hbar}{m} \tilde{\rho} \theta_x \right)_x = 0. \hfill (2.8)$$
This corresponds to
\[ \dot{q} = \frac{\hbar}{m} \rho \theta_x \] (2.9)
— which, you can check, is the same as (2.5).

### 2.2.1 Fluid analogy and Madelung

One can associate a flow velocity to a conserved quantity by writing the flux as the density times the velocity (this defines the velocity). In the current example, it follows from (2.5) that
\[ u = \frac{\hbar}{m} \theta_x \] is the probability density flow velocity. (2.10)

Introduce now \( \rho = m \tilde{\rho} \), so that \( \rho \) is the “mass probability distribution function” (with mass per unit length units). Then rewrite (2.6–2.7) in the form (see “Details” below)
\[ \rho_t + (\rho u)_x = 0, \] (2.11)
\[ (\rho u)_t + (\rho u^2 + p)_x = -\tilde{\rho} V_x, \] (2.12)

where
\[ p = \frac{\hbar^2}{2m^2} \left( \frac{R_x^2}{R} - R R_{xx} \right) = -\frac{\hbar^2}{2m^2} (\ln R)_{xx} = -\frac{\hbar^2}{4m^2} (\ln \rho)_{xx} \text{ and } R = \sqrt{\rho}. \]

Except for the strange form of the pressure, this is the same as the isentropic Euler equations of Gas Dynamics, with a probability-weighted body force \( F = -\tilde{\rho} V_x \).

1–This analogy was noted by E. Madelung: Quantentheorie in Hydrodynamischen form. Z. Phys. 40:322-326 (1926).
2–In classical mechanics the force \( -V_x \) is applied to the particle, at the particle location. Here the force is applied to the whole mass pdf, weighted by the position pdf.

Note also that, according to QM, the expected value of the particle momentum is \(-i \hbar \int \psi^* \psi_x \, dx\). This also has a fluid dynamic analog, since \(-i \hbar \int \psi^* \psi_x \, dx = \int \rho u \, dx = \text{total momentum}\).

### Details:
(2.11) is the same as (2.8). To obtain (2.12), multiply (2.7) by \( \frac{1}{m r} \), and next differentiate with respect to \( x \). This leads to
\[ u_t + u u_x - \frac{\hbar^2}{2m^2} \left( \frac{r_{xx}}{r} \right)_x = -\frac{1}{m} V_x. \]

Then multiply by \( \rho \). Upon use of (2.11), it is easy to see that this yields (2.12).

### 3 Ill posed Laplace equation problem #02

#### 3.1 Statement: Ill posed Laplace equation problem #02

Consider the following problem involving Laplace’s equation
\[ u_{xx} + u_{yy} = 0, \] (3.1)
on the annulus \( 1 < r = \sqrt{x^2 + y^2} < 2 \).

Determine \( u = h(\theta) \) on the outer boundary \( r = 2 \), given \( u = f(\theta) \) and \( u_r = g(\theta) \) on the inner boundary \( r = 1 \). Here \( \theta \) is the polar angle, and both \( f \) and \( g \) are smooth periodic functions.

Show that this is an ill-posed problem.

**Hint:** Write the problem in polar coordinates, and assume that there is a solution, \( u = u^* \), for some \( f^* \) and \( g^* \). Then consider high frequency perturbations of the form \( f = f^* + \epsilon e^{i\pi \theta} \) and \( g = g^* + \epsilon n e^{i\pi \theta} \), \( n \) an integer, where \( \epsilon \) is small and \( n \) is large.
3.2 Answer: Ill posed Laplace equation problem #02

In polar coordinates the equation takes the form
\[
\frac{1}{r} (r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0. \tag{3.2}
\]
Let \( u^*, f^*, \) and \( g^* \) be as in the hint, with
\[
h^* = u^*(2, \theta). \quad \text{Then} \quad u = u^* + \epsilon n^e \frac{e^{in \theta}}{n}, \tag{3.3}
\]
\[
h = h^* + \epsilon (2n^e + 2^{-n}). \tag{3.4}
\]
It should now be clear that the problem answer, \( h, \) does not depend continuously on the problem data \( f \) and \( g, \) hence the problem is ill-posed.

Example: let \( \epsilon = \frac{1}{4\pi} \). Then, as \( n \to \infty, f \to f^* \) and \( g \to g^*, \) but \( h \neq h^* \) (\( h - h^* \) grows without bound).

Remark 3.1 As a matter of fact, for this problem there is also an issue with the existence of a solution (i.e., there may not be one). For example, consider the case

where \( g = 0. \) Then write the (complex) Fourier series for \( f \)

Then, at least formally, the solution is given by

\[
u = f_0 + \frac{1}{2} \sum_{n \neq 0} f_n (r^n + r^{-n}) e^{in \theta} \quad \Rightarrow \quad h = f_0 + \frac{1}{2} \sum_{n \neq 0} f_n (2^n + 2^{-n}). \tag{3.6}
\]

These series do not converge unless the \( f_n \) decay exponentially with \( n. \) As a minimum \( 2^n |f_n| \to 0 \) is needed. Thus \( f \) must be very smooth, in fact analytic in the strip \( \text{Im}(\theta) < \ln 2, \) for a solution to even exist.

Remark 3.2 (Mathematical subtle point 1). To give a precise meaning to the concept of continuous (or not) dependence of \( h \) on the data \( f \) and \( g, \) we need to define what it means for two functions to be close. That is, we need a norm \( \| \cdot \|. \) Since there are many (non-equivalent) possible norms, we should worry that perhaps there is one where the argument above does not work. However, note that the perturbations satisfy

\[
|\delta g|_n = n |\delta f|_n \quad \text{and} \quad |\delta h|_n = 2^n |\delta f|_n. \quad \text{In addition, in any norm} \| \alpha F \| = |\alpha| \| F \|, \quad \text{for any scalar} \ \alpha \ \text{and function} \ F. \quad \text{Thus select} \ \epsilon = \frac{1}{n |e^{in \theta}|}. \quad \text{Then} \ |\delta f|_n = \frac{1}{n^e}, \ |\delta g|_n = \frac{1}{n}, \ \text{and} \ |\delta h|_n = \frac{2^n}{n}. \quad \text{With different input and output norms, the problem can be made well posed. For example:} \ |f|_0 = \sqrt{\sum 2^n |f_n|^2} \quad \text{and} \ |g|_0 = \sqrt{\sum 2^n |g_n|^2}, \quad \text{guarantee a solution whenever} \ |f|_1, |g|_1 < \infty. \quad \text{Then let} \ |h|_0 = \sqrt{\sum |h_n|^2}. \quad \text{However: why would the criteria for errors in the input, versus the output, be so dramatically different?}

Remark 3.3 (Mathematical subtle point 2, but with practical consequences). On the other hand, we can destroy the argument by restricting the class of allowed data functions \( f \) and \( g. \) For example, if we only allow \( f \) and \( g \) (as well as \( h \)) to have a finite number of Fourier coefficients

\[
f = \sum_{n=-N}^{n=N} f_n e^{in \theta} \quad \text{and} \quad g = \sum_{n=-N}^{n=N} g_n e^{in \theta}, \tag{3.7}
\]

with no frequency above some fixed threshold \( N, \) then the problem becomes well posed. Of course, this makes the problem into a finite dimensional one, no longer a pde problem. However, trivial as this example may be, it shows that ill-posed problems can be turned into well-posed ones by suitable restrictions/changes.

An interesting example of this occurs with CAT scans, where the objective is to reconstruct the density \( \rho = \rho(x, y) \) of a cross section of a body from the amount of damping produced on x-rays traveling across the cross section in many different directions. If one insists on recovering the point values of the density, the problem is ill-posed. However, if one only attempts to recover density local averages (a suitably ”smeared” density), the problem becomes quite reasonable. The analog of this for ”our” problem here would be to apply a ”filter” to the data \( f \) and \( g, \) so any high frequencies are removed, and then only request to recover a suitably filtered \( h. \) This, pretty much, would transform the problem into the (well posed) version above in (3.7).
4 Small transversal vibrations of a beam

4.1 Statement: Small transversal vibrations of a beam

A beam is a structure where one dimension (the axial dimension) is much larger than the other two (the transversal dimensions), see item c. In this problem you are asked to derive an equation for the small transversal vibrations of an homogeneous elastic beam with (constant) rectangular cross section, which is not under tension or compression. Further simplifying assumptions are:

a. The wavelength of the vibrations is much bigger than the transversal dimensions of the beam.

b. The vibrations are in-plane. This means that the motion of the beam is restricted to the plane determined by its axis, and the direction of one of the sides of the rectangular cross section.

Think of a blade. When the two transversal dimensions are very different, it is much harder to excite vibrations along the larger direction.

Under the conditions stated above, the beam motion can be described in terms of the position of the beam axis $y = u(x, t)$. Let $\rho = \text{constant} > 0$ be the mass per unit length of the beam. Then:

**Task #1 of 5.** Use conservation of the transversal momentum to derive an equation for $u$.

**Task #2 of 5.** Use conservation of energy to derive another equation for $u$.

**Task #3 of 5.** Show that the solutions to the task #1 equation satisfy the task #2 equation.

To do the problem you will need a few things from § 4.1.1, as follows:
— To do task #1 you need (fo.8), where $f_s$ is defined in item 5 — $E$ and $I$ are constants.
— In addition, to do task #2, you need (fo.6) and (fo.9), where $\tau_b$ is defined in item 6.
— The summary of facts in (fo.10) may be useful.

You do not need anything else, but I strongly encourage you to read, and understand, § 4.1.1.

**Hint for task #2.** Do not forget that energy flow is not only produced by forces [force times velocity], but by torque as well [torque times angular velocity]

### More tasks

**Task #4 of 5.** Show that the task #1 equation yields conservation of angular momentum.

**Task #5 of 5.** Take a thin steel blade with a rectangular cross section (e.g., the blade from a metal hand saw, see the picture). Clamp one end using a bench vise, and leave the other end free. For the equation in task #1, in this situation, what boundary conditions on $u$ should be imposed at each end?

What if there is a frictionless-hinge at each end of the blade, that keeps the end fixed, but allows the blade to freely rotate there. What boundary conditions should be used in this case?

**Hints for task #4.** (1) Recall that the angular momentum of a mass point moving in a plane is given by $A = \pm m v d$, where: $m$ is the mass; $v$ is the speed; $d$ is the distance from the straight line through the point along the direction of motion, to some fixed point in space; and the sign is positive if the mass is moving counter-clockwise around the selected point in space. For example, if the point in space is the origin, and the point path is $x = a = \text{constant}$ and $y = v t$, then $A = m a v$. To do task #4, take the selected point to be the origin of coordinates. (2) Recall also that angular momentum is produced by torque, and that torque can manifest in two ways: “directly” (as in the torque applied through the axle to a wheel), or through forces (as when you rotate a wheel by pushing through the edge): a wheel-chair can either have a motor, or the user can move the wheels with his/her hands, or both (or someone else can push the chair).

### Side remarks

6 In general the angular momentum is a vector, but for in-plane motion it is a scalar.
c. In a string the transversal dimensions are neglected (thus a string has no bending resistance). On the other hand, for a beam they are assumed small, but their effect is not neglected (thus a beam can support a transversal load).

d. An elastic beam can support vibrations along each of the two transversal directions, as well as longitudinal vibrations. It can support torsional vibrations (twist along the axis) as well. In principle it can also support torsion along the transversal directions — but these are not consistent with the beam approximation (Euler-Bernoulli assumptions, see § 4.1.1).

Euler-Bernoulli beam theory states that the beam cross sections move as rigid planes when the beam vibrates, and remain normal to the beam axis (or, in 2-D, the beam center plane, in red here). Thus the beam motion can be described in terms of the axis behavior.

Figure 4.1: Cross section of a (rectangular) vibrating beam, of width $w$ and height $h$.

The blue lines are the edge of the beam. The red line is the beam axis. The green lines are typical “fibers” — see paragraph above (fo.5). The magenta lines are typical beam cross sections, which move as rigid planes, and remain normal to the axis and fibers.

Figure 4.2: Side view of a beam undergoing in-plane motion, as per Euler-Bernoulli theory.

4.1.1 Beam elastic energy, shear force, and torque

This subsection’s purpose is to provide contextual information needed to understand and do the problem. It does not include any further tasks to be done.

For beams under deformations that are not too large (basically, the situation in item a above), the Euler-Bernoulli assumptions apply\(^7\) (see figures 4.1 and 4.2)

eb1. Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads, and can be assumed as rigid.

eb2. During deformation, the cross section of the beam remains planar, and normal to the deformed axis of the beam.

From this assumptions, and item b, it follows that we can describe the motion of the beam using just two 1D functions (see (fo.1) below, as follows:

1. Take a coordinate system such that the beam at equilibrium is $0 < x < L$, $|y| < \frac{1}{2} h$, and $|z| < \frac{1}{2} w$, where $L$ is the beam length, $h$ is its height, and $w$ is its width. Here $x$ is the axial coordinate, and the motion is in the $x$-$y$ plane (no dependence on $z$).

2. We label each mass-element in the beam by its $(x, y, z)$ coordinates at equilibrium, and describe the beam at any time by giving the coordinates of each mass-element as a function of time and the equilibrium coordinates, that is: $X = X(x, y, t)$, $Y = Y(x, y, t)$, $Z = z$, where we have used item b to simplify the dependence on $z$.

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\(^7\) These assumptions have been extensively confirmed for (solid cross section) slender beams made of isotropic materials.
3. Let \( v = v(x, t) = X(x, 0, t) - x \) and \( u = u(x, t) = Y(x, 0, t) \) be the two functions describing the motion of the beam axis. Then, from eb1 and eb2,

\[
\begin{align*}
X &= x + v - \frac{1}{d} y u_x, \\
Y &= u + \frac{1}{d} y (1 + v_x),
\end{align*}
\]

where \( d = \sqrt{(1 + v_x)^2 + u_x^2}. \) (fo.1)

This follows because the unit normal vector to each planar cross section of the beam (which moves as a rigid body, as per eb1–eb2) is given by

\[
\hat{n} = \frac{1}{d} (1 + v_x, u_x, 0)^T \quad \Rightarrow \quad \hat{t} = \frac{1}{d} (-u_x, 1 + v_x, 0)^T. \] (fo.2)

Here \( \hat{t} \) is the unit vector tangent to the planar cross section, in the \( z = 0 \) plane, pointing towards \( y > 0 \) (since \( 1 + v_x > 0 \), because \( v_x \) is small, as explained below).

Detail: \( \hat{n} \) is the unit tangent vector to the beam axis: \( X = x + v, \ Y = v, \) and \( Z = 0. \)

4. Finally, the condition of small vibrations for a beam which is not under tension/compression translates into

Both \( u_x \) and \( v_x \) are small. Furthermore: \( v_x = O(u_x^2). \) (fo.3)

Thus we neglect quadratic terms in \( u_x \), and write

\[
\begin{align*}
X &= x - y u_x \quad \text{and} \quad Y = y + u. 
\end{align*}
\] (fo.4)

The assumption here is that the deviations of the beam shape from horizontal and straight are small. Under this condition, in the absence of (significant) axial stretching or compression, the changes in horizontal dimensions of the beam cannot be larger than quadratic in \( u_x. \)

Because the beam cross sections do not deform, and behave as rigid surfaces, all the deformation occurs along the curves \( y = \text{constant} \) and \( z = \text{constant}, \) which are either stretched or compressed. We can thus obtain the elastic energy in the beam by computing the elastic energy in each fiber, and integrating over all of them. Along each fiber the change in arclength (relative to equilibrium) is

\[
\Delta L ds = \left( \sqrt{X_x^2 + Y_x^2} - 1 \right) dx = -y u_{xx} dx, \]

upon use of (fo.3–fo.4). From remark 4.1, \( \frac{1}{2} E I \int dz \int dy y^2 \int u_{xx}^2 dx \) is the elastic energy in each fiber. Thus

\[
Y = \frac{1}{2} E I \int u_{xx}^2 dx = \text{beam elastic energy,} \quad \text{where} \quad I = \frac{1}{12} h^3 w \] (fo.6)

is the second moment of the beam’s cross section \( I = \iint y^2 dz dy. \)

Next we will use (fo.6) to compute both the shear force, \( f_s, \) and the torque, \( \tau_0, \) along the beam:

5. At any point along the beam, \( f_s = f_s(x, t) \) is the force in the \( y \)-direction (transversal) that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite force is applied by the right section on the left section). If the beam were to be cut, these are the forces that would be needed to keep in position the lips of the cut.

\[\footnote{This is an approximation. There is some deformation and there are forces. But they are small and we neglect them.} \]

\[\footnote{We will call these curves “fibers”} \]
6. At any point along the beam, $\tau_b = \tau_b(x, t)$ is the torque that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite torque is applied by the right section on the left section). If the beam were to be cut, these are the torques that would be needed to keep the beam ends at the cut from rotating.

Let us now investigate how $V$ changes as we deform the beam, from some configuration to another. To do this we need to apply forces to the beam, which do work against the opposite forces generated by the elastic deformation of the beam (this is how the energy changes). Thus, by looking at how the energy changes as the beam is deformed, we can ascertain the elastic forces in the beam for any given configuration.

Hence assume that $u = u(x, \tau)$, where $\tau$ is a parameter that we use to describe the successive configurations of the beam as its shape changes. Then, for any arbitrary interval $a < x < b$, the energy within the interval varies as follows

$$\frac{dV}{d\tau} = EI \int_a^b u_{xx} u_{xxt} \, dx$$

(integrate by parts twice)

$$= EI \left( u_{xx} u_{xxt} \right)_a^b - EI \left( u_{xxx} u_{x} \right)_a^b + EI \int_a^b u_{xxxx} u_{x} \, dx.$$  \hspace{1cm} (fo.7)

This equation tells us that

7. The terms $EI u_{xxx} u_x$ at $x = a$, and $-EI u_{xxx} u_x$ at $x = b$, are the work (per unit $\tau$-time) done by the applied forces to move the ends of the beam at the velocity $u_x$ of each end. This means that forces $EI u_{xxx}$ and $-EI u_{xxx}$ must be applied there. These must be the forces applied by the beam regions on each side of the interval. We conclude that

$$f_s = EI u_{xxx}.$$ \hspace{1cm} (fo.8)

8. Since $u_{x}$ is the rate of change in beam angle, and work is also torque times angle, an argument entirely similar to the one in 7 shows that

$$\tau_b = -EI u_{xx}.$$ \hspace{1cm} (fo.9)

9. The last term in (fo.7) yields the force per unit length that must be applied along the beam to cause the deformation: $-EI u_{xxxx}$. The force per unit length needed to keep the beam with the given shape, and equilibrate the force $EI u_{xxxx}$ with which the beam pushes back.

Note that the forces producing beam deformation need not be “external forces” applied to the beam. When the beam is vibrating, the forces involved are those caused by the inertia of the beam.

**Summary of useful facts.**

- $u$ is the beam transversal deformation. The beam axis is $y = u(x, t)$ (small vibrations).
- $u_t$ is the beam transversal velocity.
- $u_x$ is the beam angle.
- $u_{xt}$ is the beam angular velocity.
- $\tau_b$ is the torque (see item 6), given by $\tau_b = -EI u_{xx}$.
- $f_s$ is the shear force (see item 5), given by $f_s = EI u_{xxx}$.

The elastic energy density is $\frac{1}{2} EI u_{xx}^2$.

Remark 4.1 What is the energy stored in a slightly stretched/compressed elastic thin string?

The calculation below is valid as long as Hooke’s law applies. In addition, we neglect any changes in the area of the string cross section when under tension or compression. In fact, the Euler-Bernoulli assumptions require that changes of this type be ignored.

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$^{10}$The idea is that we deform the beam very, very, slowly. Thus, at every moment the applied forces, and the elastic forces generated by the deformation equilibrate each other exactly. There is no kinetic energy; all the work done by the applied forces go into the beam’s elastic energy. Thus $\tau$ is a very “special” time.
First consider the energy stored in a short straight segment of length \( L(t) \), as we stretch/compress it from length \( L_0 = L(0) \) to \( L_1 = L(T) \), where \( L_0 \) is the equilibrium length. From Hooke’s law, the elastic force is
\[ F = E \frac{L(t) - L_0}{L_0} A, \]
where \( A = \text{area of string cross section} \), (fo.11)
and \( E = \text{Young’s modulus} \) (\( E \) has units of force over area). The energy at the end of the process is
\[ \mathcal{V}_{se} = \int_0^T F(t) \frac{dL}{dt} \, dt = \frac{1}{2} E A \frac{(L_1 - L_0)^2}{L_0}. \]
(fo.12)
Note that this formula depends only on \( L_1 \) and \( L_0 \), not how we go from one to the other. Think now of the whole string as composed of a sequence of infinitesimal straight segments. Then we can use (fo.12) to write
\[ \mathcal{V}_{se} = \frac{1}{2} E A \int \left( \sqrt{X^2 + Y^2} - 1 \right)^2 \, ds, \]
(fo.13)
where points on the string is labeled by \( s \) (the arclength along the string at equilibrium), the string is given by \( x = X(s) \) and \( y = Y(s) \), and the dots indicate derivatives with respect to \( s \).

Claim: Take, in (fo.12), \( L_0 = ds \). Then \( L_1 = \sqrt{X^2 + Y^2} \, ds \), and \( \mathcal{V}_{se} = \frac{1}{2} E A \left( \sqrt{X^2 + Y^2} - 1 \right)^2 \, ds \).

4.2 Answer: Small transversal vibrations of a beam

With the approximations used, the transversal momentum density (momentum per unit length) is \( \rho u_t \). The momentum flux is given by the shear force \( f_s \) in (fo.8). Conservation yields
\[ (\rho u_t)_t + (EI u_{xx})_x = 0. \]
Homogeneous beam equation. (4.1)

Conservation of energy. Equation (fo.6) yields the potential (elastic) energy. The energy density is \( e_d = \frac{1}{2} \rho u_t^2 + \frac{1}{2} E I u_{xx}^2 \). The energy flux is provided by the work per unit time done by the shear force, \( f_s u_t \), and the work done per unit time by the torque, \( \tau_b u_{xt} \) (note that \( u_x = \tan \theta = \theta \) in this linear approximation, thus \( \theta_t = u_{xt} \)). Putting this all together gives the conservation of energy equation
\[ \left( \frac{1}{2} \rho u_t^2 + \frac{1}{2} E I u_{xx}^2 \right)_t + \left( E I u_{xxx} u_t - E I u_{xx} u_{xt} \right)_x = 0. \]
(4.2)
By direct differentiation it is easy to see that the solutions to (4.1) satisfy (4.2).

One may ask the question: why is it that conservation of energy does not yield a new equation? The answer (or, at least, one answer) is that, in the situation being considered there is no mechanism for energy transfer between mechanical and internal energy. There are no losses of mechanical energy. Of course, in the real world a beam does not vibrate for ever (unless energy is continuously supplied to it). The vibrations are damped, and eventually become heat.

Conservation of angular momentum. If we use the origin as the center for the angular momentum, the angular momentum density is \( x \rho u_t \). The angular momentum flux is given by \( \tau_b \), and the torque produced by the shear force: \( x f_s \). The conservation of angular momentum gives the equation
\[ (x \rho u_t)_t + (EI x u_{xxx} - EI u_{xx})_x = 0. \]
(4.3)
It is easy to see that this is the same as (4.1).

Note. Let \( \vec{r} = (X, Y, z) \) be the position vector for a mass-element in the beam — as given by (fo.4) for each fixed \( x, y, z \). Then the angular momentum of the element is
\[ \mathbf{m}_{op} = \mu \vec{r} \times \vec{v} \, dx \, dy \, dz, \]
where \( \vec{v} = \frac{d}{dt} \vec{r} = (-y u_{xt}, u_t, 0) \) is the parcel velocity and \( \mu \) is the mass-density (mass per unit volume) of the beam material — assumed constant. To be consistent with prior approximations, we should neglect the longitudinal component of the velocity, and write \( \vec{v} \approx (0, u_t, 0) \). Thus \( \mathbf{m}_{op} \approx \mu (z u_t, 0, X u_t) \, dx \, dy \, dz \). Upon neglecting the nonlinear terms as well, this yields \( \mathbf{m}_{op} \approx \mu (z u_t, 0, x u_t) \, dx \, dy \, dz \). Finally,
integrating over the beam cross-section — \( |z| \leq \frac{1}{2} w \) and \( |y| \leq \frac{1}{2} h \) — we obtain the angular momentum density \((0, 0, \rho_x u_t)\).

This has a single non-zero component, whose conservation yields (4.3).

Note that, if nonlinear terms are neglected, but the longitudinal component of the velocity is kept, the result is then \( m_{op} \approx \rho x u_t + \mu \int \mu I u_{xt} \) dx dy dz. Upon integration over the beam cross-section, this yields the non-zero component \( \rho_x u_t + \mu I u_{xt} \). The extra term here can be interpreted as the spin density \( s = \mu I u_{xt} \) (at least to the extent that keeping higher order here, when neglected elsewhere, makes sense). It turns out that (4.1) also yields conservation of the spin: \( s_t + (\frac{E I^2}{A} u_{xxxx})_x = 0 \) — however, the meaning/source of the flux here is not clear. Is there one?

Note that, in the energy equation the energy due to spin is neglected. The corresponding kinetic energy density is \( s_e = \int \frac{1}{2} \mu y^2 u_{xt}^2 dy dz = \frac{1}{2} \mu I u_{xt}^2 \). Equation (4.1), again, yields a conservation form: take its \( x \)-derivative and then write the equivalent of (4.2). Is there a meaning to this?

Finally, a way to make consistent sense of all the approximations is: Not only are we looking at the linear limit of small oscillations, but also at the “long wave limit”, which in this problem amounts to \( \partial_x = O(\epsilon) = \text{small and} \ \partial_t = O(\epsilon^2) \). Hence, for example, \( u_{xt} \) should be neglected in comparison with \( u_t \).

Boundary conditions. Using (fo.10) we see that:

- At a clamped end both \( u \) and \( u_x \) are prescribed — e.g., \( u = u_x = 0 \).
- At a free end the torque and the shear should vanish, thus \( u_{xx} = u_{xxx} = 0 \).
- At a hinged end there should be no torque, but position is given — e.g., \( u = u_{xx} = 0 \).

A situation where \( u_x = u_{xxx} = 0 \) at an end could be devised as follows: have a vise that can freely (no friction) slide up and down a vertical rod at, say, \( x = 0 \). Then clamp the blade to the vise. In this case there is no shear at the end, so that \( u_{xx} = 0 \), and \( u_x = 0 \) because of the clamping. But neither the torque, nor the position of the beam are prescribed there.

Challenge questions:
Is it possible to design a situation such that \( u = u_{xxx} = 0 \) at an end? How about \( u_x = u_{xx} = 0 \)?

5 The leaky bucket

5.1 Statement for “The leaky bucket”

This is a problem from the book by Strogatz: Nonlinear dynamics and chaos.

The following example\(^ {11} \) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see a water bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying\(^ {12} \) a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model for the situation. Let \( h(t) \) = height of the water remaining in the bucket at time \( t \); \( a \) = area of the hole; \( A \) = cross-sectional area of the bucket (assumed constant); \( v(t) \) = velocity of the water passing through the hole.


\(^ {12} \) Note that, in this problem, evaporation effects are neglected.
a. Show that $a v(t) = A \dot{h}$. What physical law are you invoking? **Warning:** since $\dot{h} < 0$, this presumes that we assign a negative value to the velocity $v$. This is a weird choice, implicit in the problem statement, but acceptable.

b. To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount $\Delta h$, and that the water has density $\rho$. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2 g h$ — $g =$ gravity acceleration.

c. Combining a and b, show that $\dot{h} = -C \sqrt{h}$, where $C = \frac{a}{A} \sqrt{2 g}$.

d. Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique in backwards time, i.e., for $t < 0$.

The description/derivation above ignores surface tension. Briefly discuss the effect surface tension has on the outcome.

5.2 Answer for “The leaky bucket”

a. Since water is conserved (conservation of mass), the rate at which the water leaves the bucket — i.e.: $a v(t)$, must equal the rate at which the water in the bucket decreases — i.e.: $A \dot{h}$. Hence: \[ a v(t) = A \dot{h} \] **Warning:** since $\dot{h} < 0$, this presumes that we assign a negative value to the velocity $v$. This is a weird choice, implicit in the problem statement, but acceptable.

b. The potential energy of the water in the bucket is given by $V = \int_0^h g A \rho y dy = \frac{1}{2} g A \rho h^2$. The rate at which kinetic energy is transported out of the bucket by the escaping water is $\dot{K} = \frac{1}{2} m \dot{v}^2$, where $m = \rho A \dot{h}$ is the rate at which mass leaves the bucket. Hence $\dot{K} = \frac{1}{2} \rho A \dot{h} v^2$. We now invoke conservation of energy, neglecting frictional losses, and equate $\dot{V} = \dot{K}$ leading to $v^2 = 2 g h$.

Note that an alternative expression for the rate at which kinetic energy is transported out of the bucket by the escaping water is given by $\dot{K} = \frac{1}{2} \rho a v^3$ — since in an infinitesimal time interval $dt$ the kinetic energy transported out is $dK = \frac{1}{2} (\rho a v dt) v^2$. Using a, it is easy to see that this is equivalent to the expression derived in the prior paragraph.

c. A simple calculation using a and b (be careful with the signs, recall that: $v < 0$ and $\dot{h} < 0$) then shows that $\dot{h} = -C \sqrt{h}$, where $C = \frac{a}{A} \sqrt{2 g} > 0$, and the implicit restriction on the solutions given by $h \geq 0$ applies.

d. Given $h(0) = 0$ (empty bucket), then the solution(s) to the o.d.e. derived in item c are:

- Since $\dot{h} \leq 0$, and $h$ must remain non-negative, the solution is unique for $t > 0$. Namely: $h(t) \equiv 0$.
- Using separation of variables, we can find the following (infinite number of) solutions, valid for $t < 0$:

\[
    h(t) = \begin{cases} 
    0 & \text{for } t_0 \leq t \leq 0, \\
    \left(\frac{C}{A} (t_0 - t)\right)^2 & \text{for } t \leq t_0.
    \end{cases}
\]

where $t_0 \leq 0$ is arbitrary.

What about surface tension? Surface tension affects this problem in (roughly) three ways: (i) On the top surface it will create an extra force that resists the emptying of the bucket. However, by assumption, the top surface is “large”, so this is not an important effect [certainly not for a bucket-sized container]. (ii) The jet that comes out
of the hole may be fractured into droplets by surface tension. Whether or not this happens is not relevant for this problem. Once the water leaves the bucket, it no longer affects \( h \).

(iii) If the hole at the bottom is small enough, surface tension may be able to stop the flow before \( h = 0 \). What happens beyond this depends on whether or not the bucket surface is wetting. If it is not, then the interface at the hole will be pinned, with surface tension across it able to support the pressure by a nonzero \( h \) in the bucket — a stable, steady, situation. If the bucket surface is wetting, then the interface will spread, allowing more water to flow through the hole, eventually making the interface unstable. Then a drop forms and falls, re-starting the process. The bucket continues to empty out, but not through a jet, but drop-by-drop. Eventually \( h \) may be too small to support even this, and this process stops.

6 The flux for a conserved quantity must be a vector

6.1 Statement: The flux for a conserved quantity must be a vector

Note: Below (for simplicity) we present many arguments/questions in 2D. However they apply just as well in nD; \( n = 3, 4, \ldots \)

Consider some conserved quantity, with density \( \rho = \rho(\vec{x}, t) \) and flux vector \( \vec{q} = \vec{q}(\vec{x}, t) \) in 2D. Then, in the absence of sources or sinks, we made the argument that conservation leads to the (integral) equation

\[
\frac{d}{dt} \int_{\Omega} \rho \, d\vec{x}_1 \, d\vec{x}_2 = -\int_{\partial\Omega} \vec{q} \cdot \hat{n} \, ds,
\]

for any region \( \Omega \) in the domain where the conserved “stuff” resides, where \( \partial \Omega \) is the boundary of \( \Omega \), \( s \) is the arc-length along \( \partial \Omega \), and \( \hat{n} \) is the outside unit normal to \( \partial \Omega \).

There are two implicit assumptions used above

1. The flux of conserved stuff is local: stuff does not vanish somewhere and re-appears elsewhere (this would not violate conservation). For most types of physical stuff this is reasonable. But one can think of situations where this is not true — e.g.: when you wire money, it disappears from your local bank account, and reappears elsewhere (with some loses due to fees, which go to other accounts).

2. The flux is given by a vector. But: \( \Rightarrow \) Why should this be so? \( \Leftarrow \) (6.2)

The objective of this problem is to answer this question.

Given item 1, the flux can be characterized/defined as follows \( \dagger \)

For any surface element \( d\vec{S} \) (at a point \( \vec{x} \), with unit normal \( \hat{n} \))

\[
\text{the flux indicates how much stuff, per unit time and unit area, crosses } d\vec{S} \text{ from one side to the other, in the direction of } \hat{n}. \]

\( \dagger \) This is in 3D. For a 2D, change “surface element” to “line element”.

It follows that the flux should be a scalar function of position, time, and direction. That is:

\[
q = q(\vec{x}, t, \hat{n}),
\]

where \( q \) is the amount of stuff, per unit time and unit length, crossing a curve \( \dagger \) with unit normal \( \hat{n} \) from one side to the other (with direction \( \dagger \) given by \( \hat{n} \)). Then (6.1) takes the form

\[
\frac{d}{dt} \int_{\Omega} \rho \, d\vec{x}_1 \, d\vec{x}_2 = -\int_{\partial\Omega} q(\vec{x}, t, \hat{n}) \, ds.
\]

\( \dagger \) In 3D: “... and unit area, crossing a surface ...”

\( \dagger \) A positive \( q \) means that the net flow is in the direction of \( \hat{n} \).
However, in this form we cannot use Gauss’ theorem to transform the integral on the right over $\partial \Omega$, into one over $\Omega$. This is a serious problem, for this is the crucial step in reducing (6.1) to a pde.

**Your task:** Show that, provided that $\rho$ and $q$ are “nice enough” functions (e.g.: continuous partial derivatives), equation (6.5) can be used to show that $q$ has the form

$$q = \hat{n} \cdot \vec{q}, \quad (6.6)$$

for some vector valued function $\vec{q}(\vec{x}, t)$.

**Hints.**

A. It should be obvious that the flux going across any curve (surface in 3D) from one side to the other should be equal and of opposite sign to the flux in the opposite direction. That is, $q$ in (6.4) satisfies

$$q(\vec{x}, t, -\hat{n}) = -q(\vec{x}, t, \hat{n}). \quad (6.7)$$

Violation of this would result in the conserved “stuff” accumulating (or being depleted) at a finite rate from a region with zero area (zero volume in 3D), which is not compatible with the assumption that $\rho_t$ is continuous and equation (6.5). **Note:** there are situations where it is reasonable to make models where conserved “stuff” can have a finite density on curves or surfaces (e.g.: surfactants at the interface between two liquids, surface electric charge, etc.). Dealing with situations like this requires a slightly generalized version of the ideas behind (6.7).

B. Given an arbitrary small curve segment of length $h > 0$ and unit normal $\hat{n}$, realize it as the hypotenuse of a right triangle where the other sides are parallel to the coordinate axes. Then write (6.5) for the triangle, divide the result by $h$, and take the limit $h \downarrow 0$. Note that, if the segment of length $h$ is parallel to one of the coordinate axes, then one of the sides of the triangle has zero length, and the triangle has zero area — but the argument still works, albeit trivially (it reduces to the argument in A).

### 6.2 Answer: The flux for a conserved quantity must be a vector

Consider a small (straight) segment of length $0 < h \ll 1$, which is not parallel to the coordinate axes. Let $\hat{n}$ be the unit normal to the segment with a positive first component $n_1 > 0$. Assume that $n_2 > 0$. Construct a right triangle $\Omega$, with two sides parallel to the coordinate axes and hypotenuse the given segment (see figure). Then $\partial \Omega$ has outside unit normals $\hat{n}$, $-\hat{i}$, and $-\hat{j}$ (where $\hat{i}$ and $\hat{j}$ are the coordinate axes unit vectors), and equation (6.5) yields

$$O(h^2) = -q(\vec{x}, t, \hat{n}) h - q(\vec{x}, t, -\hat{i}) h n_1 - q(\vec{x}, t, -\hat{j}) h n_2 + O(h^2),$$

$$= -q(\vec{x}, t, \hat{n}) h + q(\vec{x}, t, \hat{i}) h n_1 + q(\vec{x}, t, \hat{j}) h n_2 + O(h^2), \quad (6.8)$$

where $\vec{x}$ is any point in $\Omega$ and we use (6.7) to obtain the second line from the first.

Now divide (6.8) by $h$, and take the limit $h \downarrow 0$. This yields (6.7) with

$$\vec{q} = q(\vec{x}, t, \hat{i}) \hat{i} + q(\vec{x}, t, \hat{j}) \hat{j}. \quad (6.9)$$

To complete the proof we need to consider the cases:

- Case $n_1 > 0$ and $n_2 < 0$. The argument is exactly analogous to the one above.
- Case $n_1 < 0$ and $n_2 \neq 0$. Follows from the result in (6.7).
- Case $n_1 = 0$ or $n_2 = 0$. Trivial, given $\vec{q}$ as in (6.9), and (6.7).

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15 You can assume it is straight, since a limit $h \downarrow 0$ will occur.