1 Eikonal equation

1.1 Statement: Eikonal equation

1.1.1 Introduction to the Eikonal equation (background)

Consider situations like the ones described below:

[Equation]

\[ u_t + \left(\frac{1}{2} u^2\right)_x = \delta(x - ct) \]

with \( u(x, 0) = 0 \)
A. Imagine a wave-front of some sort, propagating into a medium. For example, a combustion front, across which some chemical reactions in a solid combustible media are taking place. Then the fresh (unburnt) media is ahead of the front, and the burnt media is behind it.

If the chemical reactions happen fast enough, it is reasonable to approximate the burning region as infinitely thin, and model the front by a surface moving in space. This surface, and its evolution in time, can then be described by a function $\Phi = \Phi(\mathbf{r})$ such that the level surface

$$\Phi(\mathbf{r}) = t$$  

is the wave-front at time $t$. **Notation:** $\mathbf{r} =$ position vector in space. In 2D $\mathbf{r} = (x, y)$, and in 3D $\mathbf{r} = (x, y, z)$.

For this description to be useful we need an equation for $\Phi$, so that (for example) given that we know the position of the wave-front at some time $t_0$ (that is, we know the level surface $\Phi(\mathbf{r}) = t_0$), we can predict it for future times. We do this next.

If the media is isotropic, it is reasonable to assume that, at each point in space, the wave-front propagates normal to itself at some known velocity $c = c(\mathbf{r}) > 0$, which we can either get by direct measurement or (in principle, at least) from an analysis of the detailed physics at the burning front. With this assumption, it can be shown that **the phase function** satisfies the

**Eikonal equation:**

$$c \nabla \Phi)^2 = 1.$$  

**Note:** We have assumed here the simplest of situations, where the speed of propagation depends only on the properties of the unburnt media ahead of the front (so that $c = c(\mathbf{r})$). More complicated situations can arise, where the speed of propagation depends also on other factors (such as the local curvature of the front), or other variables (such as the local temperature) for which extra equations are needed.

B. Another circumstance under which the Eikonal equation (1.2) arises is the one where high frequency waves (say, light or sound) propagate into some media. The idea is as follows: suppose that the governing equation is the wave equation

$$u_{tt} = \text{div}(c^2 \nabla u), \quad \text{where} \quad c = c(\mathbf{r}) > 0.$$  

Consider now a stationary wave situation, where the field $u$ is vibrating everywhere with the same frequency, and all the wave-fronts move along the same path (think of the light coming from some source, which does not change in time, nor moves, and is illuminating a region of space where nothing moves or changes either). If the wave-length is short enough, then near each point in space (and time) the wave-fronts will be nearly plane and will move normal to themselves at speed $c$. In this case, if we (again) introduce a **phase function** $\Phi$ to describe the wave-fronts, equation (1.2) will apply.

Notice that in this second situation there is not a single wave-front, but many (the light source emits as many wave-fronts per unit of time as its frequency). However, all the wave-fronts behave in the same way, taking the same sequence of positions in time (with only a time delay differentiating one wave-front from the other). That is, the wave-fronts are given by the equation

$$\Phi(\mathbf{r}) = t + \zeta,$$  

where $\zeta$ is the time delay.

**Remark 1.1** The situation described in B above is the mathematical formulation of the idea that light (or high frequency sound waves) propagates along rays (which are straight on an homogeneous media, and curved otherwise). The Eikonal equation can be written in terms of characteristics (see Part II below), with the characteristics giving the rays along which light propagates.
Furthermore, it can be shown (though we will not do this here) that the energy is carried along characteristics. Thus, when the rays diverge, the energy density (light intensity) in the waves goes down. Similarly, the light intensity goes up when the rays converge. In particular, when the rays cross, an infinite energy density is predicted. Of course, we cannot take this at face value (since when rays cross, the wave-fronts develop infinite curvature, and the approximations made in deriving the Eikonal equation break down). Nevertheless, it is still true that the energy density becomes large at the locations where the rays cross.

The envelope of the rays (see Part II below) for the Eikonal equation is thus very important, since it gives the location in space where we should expect to find large wave amplitudes. It is very easy to locate these places when dealing with light, since they correspond to very bright curves and surfaces in space; known as caustics. You can easily see caustics if you put a glass filled with water, on a white surface, under a bright light.

1.1.2 Part 1, derivation of the Eikonal equation

**Derive the Eikonal equation** (1.2) to describe the situation where a moving surface, given by \( \Phi(\mathbf{r}) = t \), moves at speed \( c = c(\mathbf{r}) \) (normal to itself) at each point in space.

**Hint 1.1** Consider the wave-front at time \( t \), and let \( \hat{n} \) be the unit normal to the wave-front at any place along it (pointing into the direction of propagation). Then because the wave-front propagates normal to itself at velocity \( c \), it should be that:

\[
\mathbf{r} \text{ is in the wave-front at time } t \text{ if and only if } \mathbf{r} + \hat{n}c dt \text{ is in the wave-front at time } t + dt.
\]

In other words

\[
\Phi(\mathbf{r}) = t \iff \Phi(\mathbf{r} + \hat{n}c dt) = t + dt.
\]

(1.5)

1.1.3 Part 2, find the characteristic form and caustics

Consider the case of an homogeneous two dimensional media, where the speed of propagation \( c \) is a constant. In this case we can choose a-dimensional variables so that \( c \equiv 1 \) and the Eikonal equation reduces to

\[
(\nabla \Phi)^2 = 1,
\]

(1.6)

where \( \Phi = \Phi(x, y) \) and the wave-fronts are the curves \( \Phi(x, y) = \text{constant} \). This last equation has a characteristic form, given by:

\[
\frac{d\mathbf{r}}{dt} = \mathbf{k} \quad \text{and} \quad \frac{d\mathbf{k}}{dt} = 0, \quad \text{along which} \quad \frac{d\Phi}{dt} = 1.
\]

(1.7)

Here \( \mathbf{r} = (x, y) \) is the position vector in space and \( \mathbf{k} = \nabla \Phi \) is the wave-number vector, which is normal to the wave-fronts and, from (1.6), also a unit vector. A solution of the Eikonal equation is determined once we give a wave-front, which can then be used to provide initial values for these characteristic equations.

**Notation:** The curves in space \( \mathbf{r} = \mathbf{r}(t) \) — given by the solution of the characteristic equations (1.7) — are are called rays.

Consider now the situation where the wave-front \( \Phi = 0 \) is the parabola \( y = x^2 \). Then

(a) **Compute the characteristics for this situation.**

(b) **Consider the family of curves in space given by the rays just computed as part of doing item (a). Find the envelope of this family and express it in the form \( y = y(x) \).**

(c) **Plot the envelope computed in part (b).**

This problem has relatively simple algebra and you should not have big hassles with it, provided you are careful, organized, and keep your notation straight.

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1 You can easily get \( \hat{n} \) from \( \nabla \Phi \).
1.1.4 Hints and notation

The initial wave-front $\Phi = 0$ can be described by the parametric equations $x = s$ and $y = s^2$, where $-\infty < s < \infty$ is the parameter. The unit tangent and unit normals to this curve are

$$t = \left( \frac{1}{\sqrt{1 + 4s^2}}, \frac{2s}{\sqrt{1 + 4s^2}} \right) \quad \text{and} \quad n = \left( \frac{-2s}{\sqrt{1 + 4s^2}}, \frac{1}{\sqrt{1 + 4s^2}} \right).$$

Then for each $s$ we have a characteristic, where the formula for $n$ defines the initial value for $k$.

After you solve the characteristic equations, you will end up with formulas for the rays of the form $x = x(s, t)$ and $y = y(s, t)$, where each $s$ defines an individual ray. The rays are given parametrically, with $t$ the parameter along each ray. You should find it rather easy to eliminate the parameter $t$ from the formula for the rays, so that you can write the ray labeled by $s$ as the solution to an equation of the form

$$F(x, y, s) = 0.$$

Once you do this, you’ll have the standard form used to compute envelopes. From this, getting the envelope in the form $x = x(s)$ and $y = y(s)$ is rather straightforward. Eliminating $s$ from these two last expressions should now be easy, and it will give you the requested form $y = y(x)$ for the envelope. This will end up being a rather simple formula, that you can plot without trouble.

1.1.5 What is the envelope of a family of curves? (background)

The envelope of a family of curves can be defined for families of curves that depend smoothly on a parameter: a smooth curve for each parameter value, and the curves depend smoothly on the parameter. Specifically: The envelope of a family of curves is the locus of the points where two “infinitesimally close” members of the family intersect. Alternatively, it can also be defined as a curve $C_e$ such that: (i) Every point in $C_e$, say $p$, belongs also to a member curve of the family, say $F_p$. (ii) $C_e$ and $F_p$ are tangent at $p$.

As an example, consider the family of curves given by an equation like (1.9). Let us apply the first definition: a point $p = (x, y)$ belongs to the envelope if, for some $s$, it satisfies both

$$F(x, y, s) = 0 \quad \text{and} \quad F(x, y, s + ds) = 0.$$

But $F(x, y, s + ds) = F(x, y, s) + F_s(x, y, s)ds$. Thus the envelope curve follows from the two equations

$$F(x, y, s) = 0 \quad \text{and} \quad F_s(x, y, s) = 0,$$

(1.10)

Let us now apply the second definition: It says that we can write the envelope in the form $x = X(s)$ and $y = Y(s)$, where

$$F(X, Y, s) = 0. \quad [A]$$

and $(\dot{X}, \dot{Y})$ is orthogonal to the curve $F(x, y, s) = 0$ at $(X, Y)$ — that is:

$$\dot{X} F_x(X, Y, s) + \dot{Y} F_y(X, Y, s) = 0. \quad [B]$$

However, [A] implies

$$\dot{X} F_x(X, Y, s) + \dot{Y} F_y(X, Y, s) + F_s(X, Y, s) = 0. \quad [C]$$

Thus [B] is equivalent to

$$F_s(X, Y, s) = 0. \quad [D]$$

Note now that [A] and [D] are the same as (1.10). The two definitions give the same answer.

In the contexts where the Eikonal equation applies, the envelope of the rays (caustic) corresponds to the places where portions of the initial wave-front are squeezed into a point, so that the energy density goes to infinity. This, of course, does not quite happen in the physics (the Eikonal approximation fails at caustics), but high energy densities occur there.\(^2\)

\(^2\)It can be shown that the wave amplitude ratio (at caustic versus initial) is proportional to the inverse sixth power of the wave-length — i.e.: $r \propto \lambda^{-1/6}$. The shorter the wave-length, the bigger the magnification.
1.2 Answer: Part 1, derivation of the Eikonal equation

Following hint 1.1, we notice that the unit normal to the wave-fronts is given by
\[ \hat{n} = \frac{1}{|\nabla \Phi|} \nabla \Phi, \] (1.11)
since the gradient is always normal to the level surfaces. Furthermore, this normal points in the direction of propagation, since \( \Phi \) increases in the direction of propagation — as follows from equation (1.1), which implies that the value of \( \Phi \) along a wave-front increases as the front moves.

We now expand the second equation in (1.5) up to \( O(dt) \), and use the first equation to eliminate the leading order. This yields, upon dividing by \( dt \),
\[ (\hat{n} \cdot \nabla \Phi) c = 1 \implies c |\nabla \Phi| = 1, \] (1.12)
where we have used that \( \nabla \Phi = |\nabla \Phi| \hat{n} \) (as follows from (1.11)) to get the second equality. The second equation here is clearly equivalent to the Eikonal equation (1.2), since \( c > 0 \).

1.3 Answer: Part 2, find the characteristic form and caustics

Consider the Eikonal equation for a 2-D media with constant wave speed \( c = 1 \). That is
\[ (\nabla \Phi)^2 = 1, \quad \text{where} \quad \Phi = \Phi(x, y), \] (1.13)
and the wave-fronts are given by \( \Phi(x, y) = t = \text{constant} \). The characteristic equations for (1.13) are
\[ \frac{dr}{dt} = k \quad \text{and} \quad \frac{dk}{dt} = 0, \quad \text{along which} \quad \frac{d\Phi}{dt} = 1, \] (1.14)
where \( t = \text{time}, \ r = (x, y) \) is the position vector in space and \( k = \nabla \Phi \) is the wave-number vector. Note that \( k \) is normal to the wave-fronts and (from (1.13)) it is also a unit vector.

Let the initial wave-front \( \Phi = 0 \) be the parabola \( y = x^2 \), parametrized by \( x = s \) and \( y = s^2 \), where \( -\infty < s < \infty \). The initial conditions for the characteristic equations are then:
\[ r = (s, s^2), \quad k = \left( \frac{-2s}{\sqrt{1 + 4s^2}}, \frac{1}{\sqrt{1 + 4s^2}} \right) \quad \text{and} \quad \Phi = 0 \quad \text{for each} \quad -\infty < s < \infty. \] (1.15)
The solution to the characteristic equations (1.14) is then \( \Phi = t, \ k \equiv \text{constant} \) and
\[
\mathbf{r} = (x, y) = (s, s^2) + \left( \frac{-2s}{\sqrt{1 + 4s^2}}, \frac{1}{\sqrt{1 + 4s^2}} \right) t.
\]
(1.16)

From the expression here for \( y \) along each ray, it is clear that \( t = (y - s^2) \sqrt{1 + 4s^2} \). Using this to eliminate \( t \) in the expression for \( x \) in (1.16), we obtain the following equation for the rays
\[
F = F(x, y, s) = x + 2sy - s - 2s^3 = 0.
\]
(1.17)

The envelope for the family of curves (rays) given by (1.17) follows from \( F = \frac{\partial F}{\partial s} = 0 \). The equation \( \frac{\partial F}{\partial s} = 0 \) yields \( y = 3s^2 + \frac{1}{2} \), which can be used in (1.17) to eliminate \( y \) and obtain the envelope in parametric form:
\[
x = -4s^3 \quad \text{and} \quad y = 3s^2 + \frac{1}{2} \quad \Rightarrow \quad y = \frac{1}{2} + \frac{3}{\sqrt{16}} x^{2/3}.
\]
(1.18)

Clearly the envelope has a cusp at \( \mathbf{r} = (0, \frac{1}{2}) \) — see figures 1.1 and 1.2.

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**Figure 1.2**: Envelope of the rays for the Eikonal equation (1.13), for a parabolic initial wave-front. The envelope is given by equation (1.18).

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## 2 Gas Dynamics (Eulerian to Lagrangian formulation)

### 2.1 Statement: Gas Dynamics (Eulerian to Lagrangian formulation)

The (inviscid) Euler equations of Gas Dynamics, in one space dimension, have the following form in the laboratory (Eulerian) frame:\(^3\)
\[
\begin{align*}
\rho_t + (\rho u)_x &= 0 \quad \text{(conservation of mass)}, \\
(\rho u)_t + (\rho u^2 + p)_x &= 0 \quad \text{(conservation of momentum)}, \\
(\rho E)_t + (\rho E u + p u)_x &= 0 \quad \text{(conservation of energy)},
\end{align*}
\]
(2.1)

where \( \rho = \rho(x, t) \) is the mass density, \( u = u(x, t) \) is the flow velocity, \( p = p(x, t) \) is the pressure, \( E = \frac{1}{2} u^2 + e \) is the energy per unit mass, and \( e \) is the internal energy per unit mass — given by an equation of state \( e = \mathcal{E}(\rho, p) \). Of course, \( x \) is distance and \( t \) is time.

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\(^3\)In the absence of body forces, such as gravity.
In the special case of adiabatic (constant entropy) motion, we have $0 = T dS = de - pdv$, where $T$ is the temperature, $S$ is the entropy, and $v = 1/\rho$ is the specific volume. Then the equation of state specifies $p$ as a function of $\rho$, and the system can be reduced to the first two equations above — the conservation of energy equation is not needed.

Introduce the (mass) Lagrangian coordinate $\zeta = \zeta(x, t)$ by

$$\zeta = \int_{x^*}^{x} \rho(s, t) \, ds,$$  \hspace{1cm} (2.2)

where $x = x^*(t)$ is the position of some (arbitrary) point in the gas — i.e.: $\frac{dx^*}{dt} = u(x^*, t)$. Then, as long as no vacuum state arises,\(^4\) the transformation from the Eulerian coordinates $(x, t)$ to the (mass) Lagrangian coordinates $(\zeta, t)$ has an inverse, given by

$$x = x^* + \int_{0}^{\zeta} v(z, t) \, dz,$$  \hspace{1cm} (2.3)

where we think of $v$ as a function of $\zeta$ and $t$ inside the integral — i.e.: $v = v(\zeta, t)$.

Assume now that no vacuum state arises, and that the solutions are piece-wise smooth (i.e.: shocks, for example, are allowed). Then

1. Prove the formula in (2.3) for the inverse.
2. Let $Z = Z(x, t)$ be the function defined by the right hand side in (2.2) — i.e.: $\zeta = Z(x, t)$. Show that

$$Z_x = \rho \quad \text{and} \quad Z_t = -\rho u.$$  \hspace{1cm} (2.4)

In particular: $Z_t + u Z_x = 0$, so that $Z$ is constant along the particle paths — this shows that, indeed, $\zeta$ is a Lagrangian coordinate.

3. Let $X = X(\zeta, t)$ be the function defined by the right hand side in (2.3) — i.e.: $x = X(\zeta, t)$. Show that

$$X_\zeta = v \quad \text{and} \quad X_t = u.$$  \hspace{1cm} (2.5)

Again: $\zeta = \text{constant}$ should be a particle path, hence its Eulerian coordinate must move at the flow speed.

4. Transform coordinates, from Eulerian to Lagrangian, in the equations given by (2.1). Show that, in Lagrangian coordinates, the equations have the conservation form

$$\begin{align*}
v_t - u_\zeta &= 0 \quad \text{(conservation of volume),} \\
u_t + p_\zeta &= 0 \quad \text{(conservation of momentum),} \\
E_t + (pu)_\zeta &= 0 \quad \text{(conservation of energy).}
\end{align*}$$  \hspace{1cm} (2.6)

Notice how much more compact than in Eulerian coordinates the equations are.

5. Derive directly, using conservation arguments, the equations in (2.6).

**Hints regarding item 4.** It is when doing item 4 that you will have to pay particular attention to the fact that derivatives may fail to exist in the classical sense. The usual ways in which coordinate transformations are carried will not remain valid when discontinuities in the solution arise. I encourage you to (first) carry the transformation in the usual way (assuming that all the functions involved have derivatives), to convince yourself that the calculations cease to make sense when discontinuities arise. For example, what is the meaning of stuff like $u p_x$ when both $u$ and $p$ have a discontinuity at some point $x = s$? In this case $p_x$ will have a Dirac’s delta function contribution at $x = s$ and, since $u$ has no unique value at $x = s$, the product $u p_x$ has no meaning.

\(^4\)So that $\rho > 0$ everywhere.
In order to avoid the difficulties pointed out in the prior paragraph (when discontinuities in the solution are present) you should go back to the integral formulation of the conservation laws,\(^5\) and transform them directly. Namely, instead of (2.1), use:

\[
\frac{d}{dt} \left( \int_a^b \rho \, dx \right) = \left( (u - \dot{a}) \rho \right)_{x=a} - \left( (u - \dot{b}) \rho \right)_{x=b} \quad \text{(cons. of mass)},
\]

\[
\frac{d}{dt} \left( \int_a^b \rho \, u \, dx \right) = \left( (u - \dot{a}) \rho \, u + p \right)_{x=a} - \left( (u - \dot{b}) \rho \, u + p \right)_{x=b} \quad \text{(cons. of momentum)},
\]

\[
\frac{d}{dt} \left( \int_a^b \rho \, E \, dx \right) = \left( (u - \dot{a}) \rho \, E + p \, u \right)_{x=a} - \left( (u - \dot{b}) \rho \, E + p \, u \right)_{x=b} \quad \text{(cons. of energy)},
\]

for any interval \(a = a(t) < x < b = b(t)\) — and transform these into Lagrangian coordinates. You should find out that the conservation of mass transforms into a trivial equation in Lagrangian coordinates. On the other hand, the conservation of volume, which is trivial in Eulerian coordinates:

\[
\frac{d}{dt} \left( \int_a^b dx \right) = -\dot{a} + \dot{b},
\]

yields a non-trivial equation in Lagrangian coordinates.

### 2.2 Answer: Gas Dynamics (Euler to Lagrangian formulation)

1. For any fixed time, \(\zeta\) is a strictly increasing function of \(x\), hence it has a (unique) inverse. Furthermore \(\frac{d\zeta}{dx} = \rho\), with \(\zeta = 0\) for \(x = x^*\). Hence \(\frac{dx}{d\zeta} = v\), with \(x = x^*\) for \(\zeta = 0\) — from which the formula for the inverse transformation in (2.3) follows. Shocks are not a problem for this argument, since it involves no differentiations across discontinuities — Of course: \(\zeta\) is always a continuous function of \((x, t)\), and \(x\) is always a continuous function of \((\zeta, t)\).

2. Let \(Z = Z(x, t)\) be the function defined by the right hand side in (2.2). It is then obvious that \(Z_x = \rho\). On the other hand

\[
Z_t = \int_{x^*}^x \rho_t \, dx - \rho(x^*, t) \frac{dx^*}{dt} = -\int_{x^*}^x (\rho \, u) \, dx - \rho(x^*, t) \, u(x^*, t) = -\rho(x, t) \, u(x, t).
\]

This finishes the proof of (2.4). Again, the argument works even if shocks are present, because the differentiation across (possible) discontinuities occurs under the integral sign, and then the equality \(\rho_t = -(\rho \, u)_x\) — true even for weak solutions — is used.

Alternatively: \(Z\) is the total mass between \(x^*\) and \(x\). The mass flux across \(x^*\) is zero — since \(x^*\) is a particle path — while the mass flux across \(x\) is \(\rho \, u\). Hence, conservation of mass yields \(Z_t = -\rho \, u\).

3. Let \(X = X(\zeta, t)\) be the function defined by the right hand side in (2.3). Then \(X(\zeta, t) = x\), where \(\zeta = Z(x, t)\). Thus \(X_z \, Z_x = 1\) and \(X_z \, Z_t + X_t \, Z_x = 0\). From this, using equation (2.4), equation (2.5) follows.

4. Here we transform from Euler to Lagrangian coordinates the system in (2.1). We first use an argument that ignores the (possible) presence of discontinuities — i.e.: assume that all the functions involved have derivatives in the usual “strong” sense.

First, from equation (2.4), we see that derivatives transform as follows:

\[
\begin{align*}
\partial_x & = \rho \partial_x^L & \text{and} & \quad \partial_t & = \partial_t - \rho \, u \, \partial_x^L & \Rightarrow & \quad \partial_t + u \, \partial_x = \partial_x^L \quad \text{(2.8)}
\end{align*}
\]

\(^5\) Which remain valid even for discontinuous solutions.
Next, rewrite the system in (2.1) in the form
\[
\begin{aligned}
\rho_t + u\rho_x &= 0, \\
u_t + uu_x &= 0, \\
E_t + uE_x &= 0.
\end{aligned}
\] (2.9)

Using now (2.8) to transform (2.9), we obtain (2.6):
\[
\begin{aligned}
0 &= \rho_t + \rho^2 u\zeta = -\rho^2 (v_t - u\zeta), \\
0 &= u_t + pu\zeta, \\
0 &= E_t + (pu)\zeta.
\end{aligned}
\] (2.10)

In order to obtain a derivation that works for weak solutions — with shocks, etc., we go back to the integral formulation of the conservation laws. 7 Consider an “arbitrary” interval \(a(a(t) < x < b = b(t))\) in the laboratory (Eulerian) coordinates, and the corresponding interval \(A = A(t) = Z(a, t) < \zeta < B = B(t) = Z(b, t)\) in the Lagrangian frame. In particular, note that (2.4) implies that
\[
\dot{A} = (\rho \dot{a} - \rho u) \zeta = A \quad \text{and} \quad \dot{B} = (\rho \dot{b} - \rho u) \zeta = B.
\] (2.11)

4.a — The conservation of mass in the Eulerian frame has the form:
\[
\frac{d}{dt} \left( \int_a^b \rho \, dx \right) = \left( (u - \dot{a}) \rho \right)_{x=a} - \left( (u - \dot{b}) \rho \right)_{x=b}.
\] (2.12)

Using now that \(\rho \, dx = d\zeta\), and (2.11), this becomes
\[
\frac{d}{dt} \left( \int_A^B d\zeta \right) = \left( (u - \dot{a}) \rho \right)_{\zeta=A} - \left( (u - \dot{b}) \rho \right)_{\zeta=B} = -\dot{A} + \dot{B}
\] (2.13)
in Lagrangian coordinates — which is trivial.

4.b — The conservation of volume in the Eulerian frame has the trivial form:
\[
\frac{d}{dt} \left( \int_a^b v \, dx \right) = -\dot{a} + \dot{b}.
\] (2.14)

Using now that \(\rho \, dx = d\zeta\), and (2.11), this becomes
\[
\frac{d}{dt} \left( \int_A^B v \, d\zeta \right) = -\dot{a} + \dot{b} - \left( u + \dot{A} v \right)_{\zeta=A} + \left( u + \dot{B} v \right)_{\zeta=B}
\] (2.15)
in Lagrangian coordinates. This is exactly the integral form that corresponds to the first equation in (2.6).

4.c — The conservation of momentum in the Eulerian frame has the form:
\[
\frac{d}{dt} \left( \int_a^b \rho u \, dx \right) = \left( (u - \dot{a}) \rho u + p \right)_{x=a} - \left( (u - \dot{b}) \rho u + p \right)_{x=b}.
\] (2.16)

Using now that \(\rho \, dx = d\zeta\), and (2.11), this becomes
\[
\frac{d}{dt} \left( \int_A^B u \, d\zeta \right) = \left( (u - \dot{a}) \rho u + p \right)_{\zeta=A} - \left( (u - \dot{b}) \rho u + p \right)_{\zeta=B}
\] (2.17)

\[6\] Use the first equation to eliminate time derivatives of \(\rho\) in the other two — allowed only for smooth solutions.

\[7\] From which (2.1) follows if the derivatives exist.
in Lagrangian coordinates. This is exactly the integral form that corresponds to the second equation in (2.6).

4.d — The conservation of energy in the Eulerian frame has the form:

$$\frac{d}{dt} \left( \int_{a}^{b} \rho E \, dx \right) = \left( (u - \dot{a}) \rho E + p \, u \right)_{x=a} - \left( (u - \dot{b}) \rho E + p \, u \right)_{x=b}. \quad (2.18)$$

Using now that $\rho \, dx = d\zeta$, and (2.11), this becomes

$$\frac{d}{dt} \left( \int_{A}^{B} E \, d\zeta \right) = \left( (u - \dot{a}) \rho E + p \, u \right)_{\zeta=A} - \left( (u - \dot{b}) \rho E + p \, u \right)_{\zeta=B} = \left( p \, u - \dot{A} \, E \right)_{\zeta=A} - \left( p \, u - \dot{B} \, E \right)_{\zeta=B} \quad (2.19)$$

in Lagrangian coordinates. This is exactly the integral form that corresponds to the third equation in (2.6).

5. The equations in (2.6) can be derived directly, using the standard conservation arguments in the Lagrangian frame, as follows:

Consider a tube of constant cross-sectional area $S$, in which the gas motion is taking place. Then $S \zeta$ is the total mass contained in the gas between some fixed parcel of gas — whose laboratory coordinates are given by $x = x^\ast(t)$, and the current point. In particular: $S \, d\zeta$ is the mass differential. Hence

5.a — In the Lagrangian coordinate system, $v$ is the length density — in other words: $\int_{A}^{B} v \, d\zeta$ is the distance between the gas parcels $A$ and $B$. Since length is conserved, and has flux $-u$, the first equation in (2.6) follows.

5.b — In the Lagrangian coordinate system, $Su$ is the momentum density — in other words: $\int_{A}^{B} Su \, d\zeta$ is the total momentum carried by the gas between the parcels $A$ and $B$. Since momentum is conserved, and has flux $Sp$, the second equation in (2.6) follows.

5.c — In the Lagrangian coordinate system, $SE$ is the energy density — in other words: $\int_{A}^{B} SE \, d\zeta$ is the total energy carried by the gas between the parcels $A$ and $B$. Since energy is conserved, and has flux $Sp \, u$, the third equation in (2.6) follows.

### 3 Moving point source in 1D

#### 3.1 Statement: Moving point source in 1D

Situations where there is a moving source in the context of wave propagation are common. In particular, if the source is compact and one is only interested in the resulting wave pattern far away from the source, one can often simplify the question by assuming a point source. Here we consider a simple example of this type, in 1D and for a scalar first order equation with constant coefficients (homogeneous media). We also assume “trivial” initial conditions.

---

8 Force by the pressure.
9 Work done by the pressure.
10 Distances much greater than the source size.
When the equation is linear, the problem is very simple, and the only (mildly) interesting effect that occurs is that of “resonance” when the source moves at the characteristic speed. The mathematical problem in this case is

\[ u_t + cu_x = \delta(x - st) \quad \text{and} \quad u(x, 0) = 0, \]  

(3.1)

where \( c \) is the wave speed, \( s \) is the source speed (both constants), and \( \delta(\cdot) \) is Dirac’s delta function. **Show that (3.1) is equivalent to**

\[ u_t = \delta(x - vt) \quad \text{and} \quad u(x, 0) = 0, \]  

(3.2)

for some constant \( v \). Then **solve (3.2) for all possible values of \( v \). What happens at resonance?**

The situation becomes more interesting when the equation is nonlinear. Then the source can produce (or not) a **precursor shock moving ahead of it** (depending on the source speed and strength), and the (unrealistic) growth of the linear response in the resonant case is suppressed. As an example, consider the problem for the conserved density

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = \delta(x - ct) \quad \text{and} \quad u(x, 0) = 0, \]  

(3.3)

where \( c \) is a constant. **Solve this problem for all possible values of \( c \).**

**HINTS**

**H1.** The solution responds to the delta-function forcing on the right with a **discontinuity along \( x = ct \).** The discontinuity is such that the derivatives (interpreted in the weak sense) produce the delta function. That is

\[-c [u] + \frac{1}{2} [u^2] = 1, \]  

(3.4)

where \([\cdot]\) = jump across discontinuity (value ahead minus value behind). Specifically, if \( u_a \) is the value of \( u \) immediately ahead of the discontinuity, and \( u_b \) is the value immediately behind it, then: \([u] = u_a - u_b\) and \([u^2] = u_a^2 - u_b^2\).

**Not all the solutions to this equation are acceptable.** The next hint, and remark 3.1, deal with this issue.

**H2.** Characteristics converge into shocks. However, the discontinuity along \( x = ct \) is not a shock, but the response to a point forcing: the characteristics must enter on one side of \( x = ct \), and exit on the other. The only exception is when they enter/exit on one side and are parallel on the other — see remark 3.1. But the characteristics **never converge on both sides of \( x = ct \).**

**H3.** The characteristics for the un-forced equation are: \( \frac{dx}{dt} = u \), along which \( \frac{du}{dt} = 0 \). Hence, the initial value \( u(x, 0) = 0 \) will persist at any given point \( x \), till affected by something that makes the characteristic equations fail. **That is: either a shock wave or the delta-function forcing.**

**H4.** The solution to (3.3) is rather simple. It is made up by constant strength/speed shocks, regions where \( u \) is constant, and rarefaction fans. Further: **it is a function of \( x/t \) only. Why?**

**H5.** The shock conditions for (3.3) are: (i) The shock speed is the average of \( u \) across the discontinuity. (ii) The value of \( u \) behind the shock is larger than the value ahead.

**H6.** The rarefaction fans for (3.3) are solutions where all the characteristics determining \( u \) emanate from a single point in space time and “fan” out.

**Remark 3.1** Here we elaborate on the subject matter of hints **H1** and **H2.** Specifically: **what restrictions the solutions of the jump equation in (3.4) must satisfy — which is what **H2** is all about. The objective is to understand the behavior of the characteristics for equation (3.3) at/near the location \( x = ct \) of the delta function forcing. To do this, consider (3.3) as the limit of**

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = f(\epsilon(x - ct)) \quad \text{as} \quad \epsilon \to 0, \]  

(3.5)
where $f_\varepsilon(z)$ is a smooth, positive function, with unit area, vanishing outside $|z| < \varepsilon$. The characteristic equations for this problem are

$$\frac{dx}{dt} = u, \quad \text{along which} \quad \frac{du}{dt} = f_\varepsilon(x - ct). \quad (3.6)$$

Then, as long as the characteristics are outside the forcing region $ct - \varepsilon < x < ct + \varepsilon$, they are straight lines — along which $u$ is constant. When they enter the forcing region, on the other hand, they accelerate (as $u$ increases).

Hence, the following situations (and nothing else) can arise:

$a$ Characteristic enters forcing region from the left, with $u > c$.

Then $u$ starts increasing, the characteristic speeds up and it leaves on the right side of the forcing region, carrying a larger value of $u$. The $\varepsilon \to 0$ limit of this situation is (3.7) below.

$b$ Characteristic enters forcing region from the left, with $u$ barely above $c$; i.e.: $u = c + O(\varepsilon)$.

The situation is similar to item $a$, except that the $\varepsilon \to 0$ limit is: Immediately behind $x = ct$, $u > c$ and the characteristics are parallel to the path of the delta function. Immediately ahead of $x = ct$, $u > c$ and the characteristics exit (to the right) from the path of the delta function. See (3.8) below.

$c$ Characteristic overtaken by forcing region, enters it from the right (with $u < c$), but $c - u$ sufficiently large.

Once inside the forcing region, $u$ starts increasing and the characteristic speeds up. However, before the value of $u$ along the characteristic reaches $c$, the characteristic reaches the back of the forcing region, and exits it. The $\varepsilon \to 0$ limit of this situation is (3.9) below.

$d$ Same as item $c$, but the value of $u$ (when the characteristic enters the forcing region from the right) is critical.

Then the characteristic just barely makes it out (from the back) of the forcing region. The $\varepsilon \to 0$ limit of this situation is (3.10) below.

$e$ Same as item $c$, but $u$ (when the characteristic enters the forcing region from the right) is too close to $c$.

Then, once inside the forcing region, $u$ starts increasing, the characteristic speeds up, and $u$ grows beyond $c$. Thus the characteristic ends up exiting the forcing region from the same side it entered. In the $\varepsilon \to 0$ limit of this, the characteristic “bounces back” (with a higher value of $u$) into the region ahead of the path of the delta function. But this creates a multiple valued region for the solution ahead of the delta, which means that this is an inconsistent situation, and cannot occur.\footnote{In terms of equation (3.4), this is reflected by the fact that the equation does not have a real-valued solution $u_b$ when a value $u_a < c$ that is too close to $c$ is prescribed.} In fact, what happens is that a shock wave is triggered ahead of the source, so as to prevent this situation.

Thus, in terms of $u_a$ and $u_b$ (values immediately ahead, respectively behind, the discontinuity), these are the (only) acceptable possibilities for the solutions to equation (3.4):

- Case 1: $u_a > u_b > c$. \hspace{1cm} (3.7)
- Case 2: $u_a > u_b = c$. \hspace{1cm} (3.8)
- Case 3: $u_a < u_b < c$. \hspace{1cm} (3.9)
- Case 4: $u_a < u_b = c$. \hspace{1cm} (3.10)

### 3.2 Answer: Moving point source in 1-D

1. First we consider the problem in (3.1).

Changing coordinates to a frame moving at the characteristic speed $c$, so that $x_{\text{new}} = x_{\text{old}} - ct$, the equations become those in (3.2), with $v = s - c$. Since, for $v \neq 0$, $\delta(x - vt) = \frac{1}{|v|} \delta(t - \frac{x}{v})$, the solution to (3.2) is given by:

$$\begin{align*}
\text{for } v > 0: & \quad u = 0 \text{ for } x < 0 \quad \text{and} \quad u = \frac{1}{|v|} H \left( t - \frac{x}{v} \right) \text{ for } x \geq 0,
\end{align*}$$

where $H(t)$ is the Heaviside step function.
for \( v < 0 \):
\[
\text{u} = 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad \text{u} = \frac{1}{|v|} H(t - \frac{x}{v}) \quad \text{for} \quad x \leq 0,
\]
\[
\text{for} \quad v = 0:
\[
\text{u} = t \delta(x),
\]
where \( H(x) = \frac{1}{2} (1 + \text{sign}(x)) \) is the Heaviside function.

2. Next we consider the problem in (3.3).

The characteristics for the un-forced equation are: \( \frac{dx}{dt} = u \), along which \( \frac{du}{dt} = 0 \). Hence, the initial value \( u(x, 0) = 0 \) persists at any given point \( x \), till affected by a wave that makes the characteristic equations fail: either a shock or the delta-function forcing. Following the hints, and using the results of remark 3.1 (see also remark 3.6), we find

2.1. Case \( c < 0 \). From (3.7–3.10), we see that the characteristics starting at \( t = 0 \) on \( x < 0 \) should reach the source path \( x = ct \). There equation (3.4) yields \( u = c \pm \sqrt{c^2 + 2} \) as possible values for the solution on the other side of the source path. But only \( u = c + \sqrt{c^2 + 2} \) is acceptable: corresponds to (3.7). Thus the solution is given by:

\[
\text{u} = \begin{cases} 
0 & \text{for} \quad x < ct, \\
\frac{x}{t} & \text{for} \quad 0 < x < (c - \sqrt{c^2 - 2})t, \\
\frac{c - \sqrt{c^2 - 2}}{(c - \sqrt{c^2 - 2})t} & \text{for} \quad (c - \sqrt{c^2 - 2})t \leq x < ct, \\
0 & \text{for} \quad ct < x.
\end{cases}
\] (3.11)

where there is a shock moving at speed \( s = \frac{1}{2} \left( c + \sqrt{c^2 + 2} \right) > 0 \).

2.2. Case \( c = 0 \). Then (3.11) applies as well. The situation at the delta path corresponds to (3.8).

2.3. Case \( \sqrt{2} < c \). From remark 3.1 we expect, for \( c \) large enough positive, the characteristics ahead of the delta to intersect the delta path. Large enough means \( c > \sqrt{2} \). The solution in this case, corresponding to the situation in (3.9), is

\[
\text{u} = \begin{cases} 
0 & \text{for} \quad x \leq 0, \\
\frac{x}{t} & \text{for} \quad 0 < x < (c - \sqrt{c^2 - 2})t, \\
\frac{c - \sqrt{c^2 - 2}}{t} & \text{for} \quad (c - \sqrt{c^2 - 2})t \leq x < ct, \\
0 & \text{for} \quad ct < x.
\end{cases}
\] (3.12)

Note the expansion fan that appears to the right of \( x = 0 \).

Remark 3.2 To understand how the expansion fan is generated, consider the limit used in remark 3.1. Characteristics that start to the right of the path of the forcing \( f \), (i.e.: \( x > \epsilon \) for \( t = 0 \)) enter the forcing region and go through the full width of it. In the process the value of \( u \) along them increases the full amount: from \( u = 0 \) to \( u = c - \sqrt{c^2 - 2} \). On the other hand, characteristics that start on \( -\epsilon \leq x \leq \epsilon \) at \( t = 0 \), traverse only a fraction of the forcing region, and emerge on the back side with all the values between 0 and \( u = c - \sqrt{c^2 - 2} \). As \( \epsilon \to 0 \), this later process is compressed into a single point at \((x, t) = (0, 0)\), and becomes the source of the expansion fan.

2.4. Case \( c = \sqrt{2} \). The limit \( c \downarrow \sqrt{2} \) of the solution in (3.12) applies in this case, with the constant \( u = c - \sqrt{c^2 - 2} \) region behind the forcing disappearing. The situation corresponds to (3.10).

2.5. Case \( 0 < c < \sqrt{2} \). This case corresponds to the situation described in item e of remark 3.1, with a shock ahead of the delta. The solution is

\[
\text{u} = \begin{cases} 
0 & \text{for} \quad x \leq 0, \\
\frac{x}{t} & \text{for} \quad 0 < x < ct, \\
\frac{c + \sqrt{2}}{t} & \text{for} \quad ct < x < st, \\
0 & \text{for} \quad st < x,
\end{cases}
\] (3.13)

where the shock speed is \( s = \frac{c + \sqrt{2}}{2} \), and the situation at the delta corresponds to (3.10).
Remark 3.3 To understand how the shock is generated in this case, let us go back to the argument in remark 3.2. Because $c$ is smaller now, not all the characteristics that start on $-\epsilon \leq x \leq \epsilon$ at $t = 0$ participate in producing the expansion fan. The ones that start close enough to $x = \epsilon$ have $u$ increased enough to allow them move ahead of the forcing. These are the ones that, in the $\epsilon \to 0$ limit, produce the constant $u = c + \sqrt{2}$ region ahead of the delta. This region then propagates forward (into the constant $u = 0$ region) via a shock wave.

Remark 3.4 Notice that the solution has a shock wave for all source velocities $c < \sqrt{2}$, but the shock disappears for $c \geq \sqrt{2}$. The interpretation is that the source creates a disturbance that can propagate ahead of the source if the source speed is not too large, thus a shock wave arises. On the other hand, if the source speed is large enough, all the disturbances it creates are left behind it, and no shock wave arises. In higher dimensions, a fast moving point source cannot create a shock wave that moves ahead of it, but it creates shocks that propagate to the side. In a rotationally invariant context, conical traveling shock fronts, with the source at the tip, arise (e.g., sonic boom by rocket).

Remark 3.5 Answer to the hint H4 question: why is the solution to the problem a function of $x/t$ only (see (3.11–3.13))? The reason is that [because $\alpha \delta(\alpha x) = \delta(x)$ for any constant $\alpha \neq 0$] the problem in (3.3) is invariant under stretching. In other words: if $u$ solves it, so does $u(\alpha x, \alpha t)$. Hence, assuming uniqueness of the solution, $u(x, t) = u(\alpha x, \alpha t)$ for any constant $\alpha \neq 0$. Evaluate this equality at $t = 1/\alpha = \tau$. Then $u(x, \tau) = u(x/\tau, 1)$, which proves the point (since $\tau$ is arbitrary).

Remark 3.6 The jump equation (3.4) is quite simple when written in terms of the relative velocities $U_a = u_a - c$ and $U_b = u_b - c$. That is: $U_a^2 = U_b^2 + 2$.

4 Shallow water (Energy dissipation at shocks)

4.1 Statement: Shallow water (Energy dissipation at shocks)

Consider the Shallow Water Wave equations in 1-D over a flat horizontal bottom:

$$h_t + (hu)_x = 0 \quad \text{and} \quad (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0, \quad (4.1)$$

where the first equation expresses the conservation of mass (or volume, since the density is constant), the second expresses the conservation of momentum, and $g$ is the acceleration of gravity.

Part I

For solutions without hydraulic jumps (i.e.: shocks), show that the mechanical energy is also conserved, and derive an equation of the form

$$(hE)_t + (Fe)_x = 0, \quad (4.2)$$

where $hE = \frac{1}{2}hu^2 + Pe$ is the energy density and $F = huE + W_p$ is the energy flux. \textbf{Find} explicit expressions for $Pe$ and $W_p$ — what physical interpretation do these two quantities have?

Part II

Introduce the Lagrangian coordinate $z = \int_{x^*}^x h(s, t) ds$ — where $x^* = x^*(t)$ is a point following the flow, so that $\frac{dx^*}{dt} = u^* = u(x^*, t)$ — and show that the equations take the form

$$v_t - u_z = 0 \quad \text{and} \quad u_t + \left(\frac{1}{2}gh^2\right)_z = 0. \quad (4.3)$$
These equations remain valid even when hydraulic jumps arise: the first equation expresses volume conservation in Lagrangian coordinates, while the second expresses momentum conservation.  

**Show that** in these coordinates, when hydraulic jumps are absent, the equation for the conservation of the mechanical energy takes the form

\[ E_t + (W_p)_z = 0. \]  

(4.4)

**Part III**

**Show that hydraulic jumps dissipate:** mechanical energy is lost at hydraulic jumps — the lost energy, presumably, becoming internal (thermal) energy.  Proceed as follows:

**Step III-1.** As pointed out in part II, the equations in (4.3) remain valid at hydraulic jumps. Hence, assume a hydraulic jump with constant speed \( D > 0 \), moving to the right into fluid at rest — where \( h = h_0 > 0, \ v = v_0 = 1/h_0, \) and \( u = u_0 = 0 \). Then the Rankine-Hugoniot jump conditions yield

\[ D [v] + [u] = 0 \quad \text{and} \quad -D [u] + \left[ \frac{1}{2} g h^2 \right] = 0, \]  

(4.5)

where the brackets denote the jumps across the shock of the enclosed quantities — e.g.: \([v] = v_0 - v_1\) — and a subscript 1 indicates the value of a variable behind (to the left of) the jump.

**Remark 4.1** There is no loss in generality in assuming a hydraulic jump moving to the right into fluid at rest, because the equations are: (a) Left-right reflection invariant. (b) Galilean invariant.

**Remark 4.2** Note that \( z \) has units of area, hence the shock “speed” \( D \) in Lagrangian coordinates has units of area per second. In other words: \( D \) is the volume flow per unit width across the jump.

**Remark 4.3** Notice that, because \( D \) is constant, the state behind the hydraulic jump is also constant. Thus the flow considered here is very simple: \( (h, u) = (h_0, 0) \) ahead of the shock \( (z > D t) \) and \( (h, u) = (h_1, u_1) \) behind the shock \( (z > D t) \). The conclusions, however, apply in general, since the behavior of shocks is controlled by the local values of the variables — not their derivatives.

Furthermore, the Lax entropy conditions must apply \( a_1 > D > a_0 \) — where \( a = \sqrt{gh^3} \) is the characteristic speed in Lagrangian coordinates. **Show that** this is equivalent to either of

\[ h_1 > h_0 \quad \text{or} \quad u_1 > 0. \]  

(4.6)

**Remark 4.4** The main purpose of this problem is to show that the Lax entropy conditions are exactly equivalent to the statement that hydraulic jumps dissipate. The solutions to the Rankine-Hugoniot jump conditions that do not satisfy the Lax entropy conditions create (1) mechanical energy at the hydraulic jumps. Thus they either would violate conservation of energy, or would have to transform internal (thermal) energy into mechanical energy — thus decreasing the total amount of entropy in the system, violating the second law of thermodynamics.

**Step III-2.** As you were asked to show in another problem, in the presence of a shock, equation (4.4) should be modified to

\[ E_t + (W_p)_z = -d \delta(z - D t), \]  

(4.7)
where $\delta(\cdot)$ is Dirac’s delta function, $z = D t$ is the position of the shock, and $d = D [E] - [W_p]$. The statement that **hydraulic jumps dissipate follows because $d > 0$ — show this.** This last equation shows that a point sink of mechanical energy appears at the location of the jumps.

Furthermore: **show that** for solutions of the Rankine-Hugoniot jump conditions that do not satisfy the Lax entropy condition (hence $h_1 < h_0$ and $u_1 < 0$), $d < 0$. This proves the point in remark 4.4.

**Hint 4.2** It is convenient to carry the algebra in non-dimensional variables. Use the following non-dimensional variables in your calculations: $h = \left(\frac{D^2}{g^{1/3}}\right)^{1/3} \tilde{h}$, $u = \left(\frac{D^3}{g^{1/3}}\right)^{1/3} \tilde{u}$, $E = \left(\frac{D^2}{g^{2/3}}\right)^{1/3} \tilde{E}$, and $d = \left(\frac{D^5}{g^{2/3}}\right)^{1/3} \tilde{d}$, where the variables with tildes have no dimension.

### 4.2 Answer: Shallow water (Energy dissipation at shocks)

**Part I**

For solutions without shocks\(^{15}\) the shallow water wave equations in (4.1) can be written in the equivalent form

$$\dot{h} + hu_x = 0 \quad \text{and} \quad \dot{u} + gh_x = 0,$$

(4.8)

where the dot denotes the material time derivative (rate of change along particle paths) — e.g.: $\dot{h} = h_t + u h_x$. Then, multiplying the first equation here by $\frac{1}{2} g$, the second by $u$, and adding, we obtain:

$$\dot{E} + \frac{1}{h} \left(\frac{1}{2} g h^2 u \right)_x = 0,$$

(4.9)

where $E = \frac{1}{2} (u^2 + gh)$. From this last equation it is easy to see that

$$(h E)_t + \left( h u E + \frac{1}{2} g h^2 u \right)_x = 0.$$

(4.10)

This equation expresses the conservation of mechanical energy for solutions without hydraulic jumps, since:

1. $\frac{1}{2} h u^2$ is the kinetic energy density (per unit mass) in the flow.
2. $\frac{1}{2} g h^2 = \int_0^h g y \, dy$ is the gravitational (potential) energy density (per unit mass) in the flow.
   
   Here $y$ is the vertical coordinate, with $y = 0$ at the bottom and $y = h$ at the surface.
3. $h u E$ expresses the flux of mechanical energy produced by the flow advection.
4. $\frac{1}{2} g h^2 u = \frac{1}{\rho} \int_0^h p u \, dy$ is the work (per unit mass) per unit time by the (hydrostatic) pressure $p = g \rho (h - y)$, where $\rho$ is the (constant) density of the fluid.

**Part II**

From the definition $z = \int_s^x h(s, t) \, ds$ it is easy to see that $z_t = -h u$ and $z_x = h$. Therefore the material time derivative becomes just $\partial_t$ in the Lagrangian coordinates, while $\partial_x \rightarrow h \partial_z$. Hence

5. Left equation in (4.8) $\Rightarrow \ 0 = h_t + h^2 u_z = -h^2 v_t + h^2 u_z \Rightarrow$ left equation in (4.3).
6. Right equation in (4.8) $\Rightarrow \ u_t + gh \ h_z = 0 \Rightarrow$ right equation in (4.3).
7. Equation (4.9) $\Rightarrow \ E_t + \left( \frac{1}{2} g h^2 u \right)_z = 0$, which is (4.4).

\(^{15}\) Thus the usual rules for differentiation apply.
Hence the solutions to the Rankine-Hugoniot equations — with we have dropped the tildes from the non-dimensional variables. Elimination of $[u]$ in (4.11) yields the equivalent form

$$ v_1 + \frac{1}{2} h_1^2 = v_0 + \frac{1}{2} h_0^2 \quad \text{or} \quad \frac{1}{h_1} + \frac{1}{2} h_1^2 = \frac{1}{h_0} + \frac{1}{2} h_0^2, $$

with

$$ \text{Lax entropy condition:} \quad h_1 > 1 > h_0, $$

and

$$ u_1 = \frac{h_1 - h_0}{h_1 h_0} = v_0 - v_1 = \frac{1}{2} (h_1^2 - h_0^2). $$

Multiplying equation (4.13) by $2 h_1$ yields

$$ 0 = h_1^3 - 2 \left( \frac{1}{h_0} + \frac{1}{2} h_0^2 \right) h_1 + 2 = (h_1 - h_0) \left( h_1^2 + h_0 h_1 - \frac{2}{h_0} \right). $$

Hence the solutions to the Rankine-Hugoniot equations — with $u_1$ given by (4.15) — are

8. $h_1 = h_0$ and $u_1 = 0 = u_0$. Trivial solution. There is no shock ......................... Of no interest here.

9. $h_1 = -\frac{1}{2} h_0 - \sqrt{\frac{1}{4} h_0^2 + \frac{2}{h_0}} < 0$. Physically meaningless ......................... Of no interest here.

10. $h_1 = -\frac{1}{2} h_0 + \sqrt{\frac{1}{4} h_0^2 + \frac{2}{h_0}}$. This yields $h_1$ as a strictly decreasing$^{16}$ function of $h_0$, with

$$ \lim_{h_0 \to 0} h_1 = \infty, \quad \lim_{h_0 \to \infty} h_1 = 0, \quad \text{and} \quad h_1 = 1 \text{ for } h_0 = 1. $$

From item 10 and (4.14 – 4.15), it follows that:

**The Lax entropy condition is satisfied if and only if $h_1 > h_0$ if and only if $u_1 > 0$.**

Finally, the dimensional formula $d = D [E] - [W_p] = D [E] - \left[ \frac{1}{2} g h^2 u \right]$ becomes, in terms of the non-dimensional variables

$$ d = [E] - \left[ \frac{1}{2} h^2 u \right] = \frac{1}{2} \left[ u^2 + h - h^2 u \right]. $$

Now, using the fact that $[u^2] = -u_1^2 = -\frac{1}{4} (h_1^2 - h_0^2)^2$, and $[h^2 u] = -h_1^2 u_1 = -\frac{1}{2} h_1^2 (h_1^2 - h_0^2)$, we can write

$$ d = \frac{1}{8} (h_1 - h_0) \left( h_1^3 + h_1 h_0 + h_1 h_0^2 + h_0^3 - 4 \right). $$

However, from (4.16) and item 8, $h_1^2 h_0 + h_1 h_0^2 = 2 \implies 4 = 2 h_1^2 h_0 + 2 h_1 h_0^2$, so that

$$ d = \frac{1}{8} (h_1 - h_0) \left( h_1^3 - h_1^2 h_0 - h_1 h_0^2 + h_0^3 \right) = \frac{1}{8} (h_1 - h_0)^3 (h_1 + h_0). $$

Hence, **hydraulic jumps satisfying the Lax entropy conditions have $d > 0$ \implies they dissipate mechanical energy.**

On the other hand, solutions of the Rankine-Hugoniot jump conditions that do not satisfy the Lax entropy conditions have $d < 0$, and would create mechanical energy at the jumps. **This proves the statement in remark 4.4.**

$^{16}$ It is easy to see that $dh_1/dh_0 < 0$. 
5 Weak solutions (problem #01)

5.1 Statement: Weak solutions (problem #01)

Let \( x \rightarrow f(x) \), \(-\infty < x < \infty\), be a piece-wise \( C^1 \) real valued function. That is: there is a (finite) number of points \(-\infty < x_1 < x_2 < \cdots < x_N < \infty\) at which \( f \) is not defined, and

1. \( f(x) \) has a continuous derivative in each of the intervals \( x_n < x < x_{n+1}, \) \( 0 \leq n \leq N \), where \( x_0 = -\infty \) and \( x_{N+1} = \infty \).

2. At each point \( x_n, \) \( 1 \leq n \leq N \), both the left \( f^-(x) = \lim_{x \rightarrow x_n, x<x_n} f(x) \) and right \( f^+(x) = \lim_{x \rightarrow x_n, x>x_n} f(x) \) limits are defined and finite. The derivative \( f' \) has the same property.

Using the definition of a generalized function derivative, show that

\[
f'(x) = f'_f(x) + \sum_{n=1}^{N} [f]_n \delta(x - x_n),
\]

where \( \delta \) is the Dirac’s delta function, \( [f]_n = f^+_n - f^-_n \) is the jump in the function at \( x_n \), and \( f'_f \) is the “usual” derivative of \( f \) — which is only defined for \( x \neq x_n \), and is piece-wise continuous.

Note: Assume that your test functions \( \phi \) vanish outside some finite interval, and have infinitely many derivatives. That is \( \phi \in C_0^\infty \).

5.2 Answer: Weak solutions (problem #01)

Let \( \phi \in C_0^\infty \) be an arbitrary test function. Then, using the definition of a generalized derivative (first equal sign), we have:

\[
\int_{-\infty}^{\infty} f'(x) \phi(x) \, dx = -\int_{-\infty}^{\infty} f(x) \phi'(x) \, dx = -\sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} f(x) \phi'(x) \, dx
\]

\[
= \int_{-\infty}^{x_1} f'(x) \phi(x) \, dx - f^-_1 \phi(x_1) + \sum_{n=1}^{N-1} \int_{x_n}^{x_{n+1}} f'(x) \phi(x) \, dx - f^-_{n+1} \phi(x_{n+1}) + f^+_n \phi(x_n) + \int_{x_N}^{\infty} f'(x) \phi(x) \, dx + f^+_N \phi(x_N)
\]

\[
= \int_{-\infty}^{\infty} f'_f(x) \phi(x) \, dx + \sum_{n=1}^{N} [f]_n \phi(x_n),
\]

which yields the desired result. Notice that the third equal sign follows upon integration by parts, since \( f \) has a continuous derivative in each interval.

THE END.