Robin boundary conditions and the method of images

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1 Example: Method of images for Robin BC in an interval

Here we show how to use the method of images to obtain the Green's function for the initial value problem for the heat equation, in an interval with homogeneous boundary conditions (Robin on the left, and Dirichlet on the right). *The solution thus obtained is quite complicated, and of not much use. However, this example illustrates that the principle behind the method of images remains valid if one is willing to allow for "non-standard" image singularities — see remark **1.1**.*

The problem is that of finding the Green's function, G, defined by the following problem

$$\boldsymbol{G_t = G_{xx}} \quad \text{for} \quad \boldsymbol{0 < x < 1} \;\text{ and }\; \boldsymbol{t > 0}, \quad \text{with} \tag{1.1}$$

(a) Boundary conditions $\boldsymbol{G - G_x = 0}$ at $\boldsymbol{x = 0}$ and $\boldsymbol{G = 0}$ at $\boldsymbol{x = 1}$.
(b) Initial conditions $\boldsymbol{G(x,\, 0) = \delta(x - y)}$, where $\boldsymbol{0 < y < 1}$ and $\boldsymbol{\delta =}$ Dirac's delta function.

The idea is to extend the problem to that of finding the solution in the whole real line, for a problem such that G is odd relative to $x = 1$, and $G - G_x$ is odd relative to $x = 0$. For this it is enough to produce initial data with these characteristics. Thus we propose initial data of the form

$$g_0(x) = \sum_{n=-\infty}^{\infty} \left(\phi_n(y - x + 2\,n) - \phi_n(y + x + 2\,n - 2)\right) \quad \text{for} \quad -\infty < x < \infty, \tag{1.2}$$

where the $\phi_n = \phi_n(x)$ are some appropriate "functions" to be defined below [1] — *through this argument* $0 < y < 1$ *is considered as some fixed constant.* Note that

1. **By construction, $\boldsymbol{g_0}$ is odd relative to $\boldsymbol{x = 1}$.**
2. **Assume that the $\boldsymbol{\phi_n}$ satisfy** $$\boldsymbol{\phi_n + \phi_n' = \phi_{n+1} - \phi_{n+1}'} \tag{1.3}$$
   Then $\boldsymbol{g_0 - g_0'}$ **is odd relative to $\boldsymbol{x = 0}$.**

   Proof.
   $$\begin{aligned} g_0(x) - g_0'(x) &= \sum_n \phi_n(y - x + 2\,n) + \phi_n'(y - x + 2\,n) - \phi_n(y + x + 2\,n - 2) + \phi_n'(y + x + 2\,n - 2) \\ &= \sum_n \phi_{n+1}(y - x + 2\,n) - \phi_{n+1}'(y - x + 2\,n) - \phi_{n-1}(y + x + 2\,n - 2) - \phi_{n-1}'(y + x + 2\,n - 2) \\ &= \sum_n \phi_n(y - x + 2\,n - 2) - \phi_n'(y - x + 2\,n - 2) - \phi_n(y + x + 2\,n) - \phi_n'(y + x + 2\,n) \\ &= -g_0(-x) + g_0'(-x). \end{aligned}$$

Hence we **define** the ϕ_n so that (1.3), as well as the initial condition in (1.1), apply. This leads to

3. $\boldsymbol{\phi_0(x) = \delta(x)}$.

[1] Then G will follow as the (bounded for $t > 0$) solution of the heat equation for $-\infty < x < \infty$ and $t > 0$, with the data in (1.2), restricted to $0 < x < 1$.

4. For $\boldsymbol{n \geq 0}$, $\boldsymbol{\phi_{n+1}}$ is the solution to the ode $\phi'_{n+1} - \phi_{n+1} = -\phi_n - \phi'_n$ vanishing at infinity:

$$\phi_{n+1}(x) = -\phi_n(x) + 2\,e^x \int_x^\infty e^{-s}\,\phi_n(s)\,ds \quad \Longrightarrow \quad \phi_n(x) = (-1)^n\,\delta(x) + p_n(x)\,e^x, \tag{1.4}$$

where $p_n(x) = 0$ for $x > 0$ and p_n is a polynomial of degree n for $x \leq 0$. For $x \leq 0$, p_n is defined recursively by

$$p_{n+1}(0) = (-1)^n\,2 - p_n(0) \quad \text{and} \quad p'_{n+1} = -2\,p_n - p'_n, \tag{1.5}$$

with $p_0 = 0$ — hence $p_1 = 2$, $p_2 = -4 - 4\,x$, etc.

5. For $\boldsymbol{n < 0}$, $\boldsymbol{\phi_n}$ is the solution to the ode $\phi'_n + \phi_n = \phi_{n+1} - \phi'_{n+1}$ vanishing at $-\infty$:

$$\phi_n(x) = -\phi_{n+1}(x) + 2\,e^{-x} \int_{-\infty}^x e^{s}\,\phi_{n+1}(s)\,ds \quad \Longrightarrow \quad \phi_n(x) = (-1)^n\,\delta(x) + p_n(x)\,e^{-x}, \tag{1.6}$$

where $p_n(x) = 0$ for $x < 0$ and p_n is a polynomial of degree n for $x \geq 0$. For $x \geq 0$, p_n is defined recursively by

$$p_n(0) = (-1)^{n+1}\,2 - p_{n+1}(0) \quad \text{and} \quad p'_n = 2\,p_{n+1} - p'_{n+1}, \tag{1.7}$$

with $p_0 = 0$. Clearly
$$\mathbf{p_n(x) = p_{-n}(-x)} \quad \text{and} \quad \boldsymbol{\phi_{\mathbf{n}}(\mathbf{x}) = \phi_{-\mathbf{n}}(-\mathbf{x})}. \tag{1.8}$$

Because $\phi_n(x) = 0$ for $n \geq 0$ and $x > 0$, and $\phi_n(x) = 0$ for $n \leq 0$ and $x < 0$, it follows that

$$g_0(x) = \delta(x - y) \quad \text{for} \quad 0 < x < 1. \tag{1.9}$$

Hence **the initial data in (1.2) lead to the Green function G in (1.1).** Thus, in order to find G, we need to have the solution of the heat equation with initial data $\phi_n(x)$. For $n = 0$ this is given by

$$G_0(x,\,t) = \frac{1}{2\sqrt{\pi\,t}}\,\exp\left(-\frac{x^2}{4\,t}\right). \tag{1.10}$$

For other values of n we can use the formulas that follow from the expressions in (1.4) and (1.6), as follows

For $n > 0$ use
$$G_{n+1}(x,\,t) = -G_n(x,\,t) \quad + 2\,e^x \int_x^\infty e^{-s}\,G_n(s,\,t)\,ds.$$

For $n < 0$ use
$$G_n(x,\,t) = -G_{n+1}(x,\,t) + 2\,e^{-x} \int_{-\infty}^x e^{s}\,G_{n+1}(s,\,t)\,ds.$$

The resulting expressions for the G_n are complicated. However, because of the rapid exponential decay (when t is not too large), we can write

$$G \approx \sum_{\text{small n}} G_n(y - x + 2\,n,\,t) - G_n(y + x + 2\,n - 2,\,t). \tag{1.11}$$

Remark 1.1 *The points $x = y + 2\,n$ and $x = -y - 2\,n + 2$ at which the G_n are "centered" in (1.11) are the images, upon reflection at $x = 0$ and $x = 1$, of the singularity at $x = y$ given by the delta initial condition in (1.1). The difference with standard applications of the method of images is that here, at these points, the nature of the reflected singularity is far more complex than merely a delta there — as given by the ϕ_n.*

THE END.