Lecture 24, 18.306, Tu. May 12, 2020

Point source for the wave equation
(Constant speed) \[ \text{Elliptic/hyperbolic transtion} \]

Problem \[ u_{tt} - \Delta u = \delta(x-vt) \delta(y) \delta(z) \] \[ \text{(in general} \quad \delta(x_1-vt) \delta(x_2) \cdots \delta(x_d) \text{)} \]
where \( d = \text{dimension and} \quad \Delta = \sum_{n} \partial_{x_n}^2 \)

Furthermore, assume \( 0 \leq v < 1 \)

Is there a steady state solution?

Try \[ u = \varphi(x, y, z), \quad x = \frac{x-vt}{\sqrt{1-v^2}} \]

where the factor \( \frac{1}{\sqrt{1-v^2}} \) is a useful normalization

Lorentz contraction

Then it reduces to \[ \tilde{\Delta} u = -\delta(\beta x) \delta(y) \delta(z) \]

where \( \beta = \sqrt{1-v^2} \)

and \( \tilde{\Delta} = \partial_x^2 + \partial_y^2 + \partial_z^2 \)

We now use the fact that \[ \int \beta \delta(\beta x) \varphi(x) dx = \int \delta(y) \beta \varphi(\frac{1}{\beta} y) dy = \varphi(0) \]
\[ \tilde{\Delta} u = -\frac{1}{\beta} \delta(x) \delta(y) \delta(z) \]  

In addition we want \( u \) to decay at \( \infty \) (no usual \( \rho \) from \( e \)). Then (see appendix 1) where \( \rho = \sqrt{x^2 + y^2 + z^2} \)

**Note 1:** If \( \tau = \frac{x - vt}{\beta} \) is introduced then (1) becomes

\[ u_{tt} - \tilde{\Delta} u = \frac{1}{\beta} \delta(x) \delta(y) \delta(z) \]

More generally, \( u_{tt} - \Delta u = f(x - vt, y, z) \)

In particular, note how a sphere in the transformed frame \( x^2 + y^2 + z^2 = \rho^2 \) transforms into a flattened ellipsoid in the original frame (contraction by factor \( \beta \) along direction of travel)

**Note 2:** The characteristic equation is \( \phi_t^2 - (\nabla \phi)^2 = 0 \) or \( \phi_t^2 - (\nabla \phi)^2 = 0 \), which has the solutions \( \phi = r \pm t, \ r = \sqrt{x^2 + y^2 + z^2} \), or \( \phi = \rho \pm T \).

You can check that \( \rho^2 = r^2 \) and \( r^2 = t^2 \) yield the same surface. More generally:
\[ p^2 = (\tau - \tau_0)^2 \] is the same as
\[ (x - \sqrt{\tau_0 / \beta})^2 + y^2 + z^2 = (t - \tau_0 / \beta)^2 \]
That is: the characteristics given by \( \phi \) and \( \psi \) above are "the same" modulo the position of the coordinate centers.

**Note:** Of course, (5) is valid only if the source has been active for all time! If, say, the source starts at \( t = 0 \) (with \( u = u_0 \) then), then (5) cannot be used far from the source.

**Note:** Small amplitude sound waves, or long waves in a pond, also satisfy the wave equation. However: beware of using these solutions in these contexts unless \( V < \frac{1}{c} \).
The reason is that in these systems the wave equation is valid only for small perturbations. The original systems are Galilean invariant and the wave equation is not, so the solutions diverge drastically as \( V \) approaches 1.

**What happens in 2-D?** Well then the fundamental solution does not decay at \( \infty \) and we cannot use it. Reason is that there is no steady state 1-D example. \( u_{tt} - u_{xx} = \delta(x - Vt), \quad 0 < V < 1; \) and \( u = u_t = 0 \) at \( t = 0 \).

Then the solution ahead of the \( \delta \) should be a right going wave, and the solution behind a left going wave. Thus

\[
u = \begin{cases}  f \left( \frac{t - x}{1 - V} \right) & \text{for} \quad x > Vt \\
0 & \text{for} \quad x < Vt \end{cases}
\]

\[
u = \begin{cases}  g \left( \frac{t + x}{1 + V} \right) & \text{for} \quad x > Vt \\
0 & \text{for} \quad x < Vt \end{cases}
\]
with the conditions:

\( u \) continuous at \( x=vt \) \( \therefore \ f(t) = g(t) \)

[reason for the scaling of the argument of \( f \) and \( g \) is to make them equal at \( x=vt \)].

\(-v \left[ u_x \right] = \left[ u_t \right] = 1 \)

Where \( \left[ f \right] = f_{\text{ahead}} - f_{\text{behind}} \)

This last one yields

\[ 1 = -v \left\{ -\frac{1}{1-v} f'(t) - \frac{1}{1+v} g'(t) \right\} - \left\{ -\frac{1}{1-v} f'(t) - \frac{1}{1+v} g'(t) \right\} \]

\[ = f'(t) + g'(t) \quad \text{(8a) and (8b) } \Rightarrow \]

\[ \text{For } \xi > 0 \quad f(\xi) = g(\xi) = \frac{1}{2} \xi + \text{const.} \]

However, we also want \( u = 0 \) for \( x > t \) or \( \xi < 0 \).

\( x < t \) (causality), \( \therefore \) \( f = g = 0 \) for \( \xi \leq 0 \).

\( \therefore \) \( \text{const} = 0 \) and

\[ u = \frac{1}{2(1-v)} (t-x) \quad \text{for } vt \leq x \leq t \]

\[ u = \frac{1}{2(1+v)} (t+x) \quad \text{for } -t \leq x \leq vt \]

\( 0 \) else wherever.
Note that in the limit $v \to 1$ the solution develops a discontinuity at $x = t$

$$u = \frac{1}{4} (t + x) \quad \text{for} \quad -t < x < t$$

$$u = 0 \quad \text{elsewhere.}$$

What happens if $v > 1$

$$u_{tt} - u_{xx} = \delta(x - vt)$$

$v > 1$

$u = 0$ at $t = 0$

$u_t = 0$

Left moving wave, not zero only here!
Thus

\[ u = f \left( \frac{x-t}{\sqrt{1-v^2}} \right) - g \left( \frac{x+t}{\sqrt{1-v^2}} \right) \quad \text{for } x \leq vt \]

and

\[
\begin{align*}
  f(0) &= 0 \quad \text{for } t \leq 0 \\
  g(0) &= 0 \quad \text{for } t \leq 0 \\
  f(t) &= g(t) \quad \text{for } t > 0 \\
  (u=0 \text{ at } x=vt)
\end{align*}
\]

and \( x=vt \)

\[
1 = -v [u_t] - [u_x] = \\
= v \left[ \frac{-1}{\sqrt{1-v^2}} f' - \frac{1}{\sqrt{1-v^2}} g' \right] + \left[ \frac{-1}{\sqrt{1-v^2}} f' - \frac{1}{\sqrt{1-v^2}} g' \right] \\
= -f' - g' \\
\Rightarrow \quad f' = g' = -\frac{1}{2} \xi \\
\text{for } \xi \geq 0
\]

Note: the \( \lim_{v \downarrow 0} \) gives the same answer as in (12)

Note: the initial value problem in 2 and 3D can also be done "exactly" using the Green's functions for the wave equation and Duhamel's principle, but the calculations are a bit hellish.
Let us now go back to (1) and consider the case \( v > 1 \) and \( d = 2 \). Then write

\[
U = U(T, Y) \quad \text{where} \quad T = \frac{\sqrt{v^2 - 1}}{v^2 - 1} \sqrt{t - x}
\]

\[
U_{TT} - U_{YY} = s(-\sqrt{v^2 - 1} T) s(Y) = \frac{1}{\sqrt{v^2 - 1}} s(T) s(X)
\]

We also expect that, "ahead" of the \( s \), that is \( x > vt \), it should be \( U = 0 \)

\[
\leftarrow \quad \text{i.e.} \quad T < 0
\]

We end up with an initial value problem for the wave equation

\[
U_{TT} - U_{YY} = 0
\]

where \( U = 0 \) and \( U_T \) jumps from 0 to \( 1/\sqrt{v^2 - 1} \) at \( Y = 0 \) only and it is zero elsewhere.

Solution

\[
U(T, Y) = \frac{1}{\sqrt{v^2 - 1}}
\]

\[\text{See Appendix 2}\]
Again, you can check that \( \Phi = y \pm T \) solves the characteristic equation \( \frac{\partial^2 \Phi}{\partial t^2} - (\nabla \Phi)^2 = 0 \).

Note that this solution is singular (discontinuous) at \( y = \pm T \). The solution in 3-D is harder to obtain (but it also reduces to an i.v. problem) and it is also singular (though not discontinuous) at \( \sqrt{y^2 + z^2} = T^2 \).

C Mach cone

\[
\begin{align*}
\text{light speed } c_0 & \quad \text{Cherenkov radiation} \\
\text{Media} & \quad \text{fast particle} \\
\text{light speed } c_m & \quad c_m < \sqrt{c_0}
\end{align*}
\]

Mach Cone (bright)

In nonlinear problems a shock forms and replaces the Mach cone: sonic booms.

However, remember: sonic booms, Galilean invariant.

\( \Phi \) is; Cherenkov (Lorentz invariants) similar/Not same!
Appendix 1  Fundamental solution of the
Laplace equation. \( \Delta \Phi = -8(\vec{x}) \) \( a.1 \)
where \( \Delta = \sum \delta_{ij} \). We claim that solutions are:

\( \Phi = -\frac{1}{2\pi} \ln(r) \) in 2-D \( a.2 \)

\( \Phi = \frac{1}{4\pi r} \) in 3-D and, generally \( a.3 \)

\( \Phi = \frac{1}{(d-2) \alpha_d r^{d-2}} \) for \( d \geq 3 \) \( a.4 \)

where \( r = \sqrt{\sum x_n^2} \) and \( \alpha_d = \) "area" of unit sphere.

Note: Any other solution of \( a.1 \) which is bounded
at \( \infty \) will differ from \( \Phi \) above by a constant.
The reason is Liouville's Theorem, that says
that any bounded harmonic function is constant.
The case \( d=2 \) is a bit different because then \( a.1 \)
has no solutions bounded at \( \infty \). We then require
\( \nabla \Phi \) to vanish at \( \infty \), instead of bounded at \( \infty \).
Proof. For functions of $\Gamma$ only, $\Delta \Phi = r^{1-d} (r^{d-1} \phi')'$. It is then easy to see that $\Phi$ satisfies $\Delta \Phi = 0$ for $\Gamma \neq 0$. Thus we only need to check the behavior at $\Gamma = 0$. Now, for any test function $\Phi$, by definition it must be $\dagger$

$$\Phi(0) = -\int \Phi(\Delta \Phi) \Phi = -\int \Phi \Delta \Phi = \int r^{d-1}$$

$$= -\alpha_d \int_0^\infty \Phi r^{1-d} (r^{d-1} \Phi')' dr =$$

$$= -\int_0^\infty \frac{1}{(d-2) r^{d-2}} (r^{d-1} \Phi')' dr$$

$$= \left[-r \Phi' \right]_0^\infty - \int_0^\infty \Phi' dr = \Phi(0) \checkmark$$

The calculation for $d = 2$ is a bit different, but works the same way.

$\dagger$ Here I assume $\Phi = \Phi(\Gamma)$ for simplicity. But, if not, note $\int (\Delta \Phi) \Phi = \int (\Delta \Phi) \bar{\Phi}$, where $\bar{\Phi} = \bar{\Phi}(\Gamma)$ is the average of $\Phi$ over the angular variables.
Appendix 2  Here we verify that the solution described in the figure at the bottom of page 29.8 solves
\[ u_{tt} - u_{yy} = \frac{1}{\beta} \delta(t) \delta(x) \]  \( \text{(A.6)} \)
with \( u \equiv 0 \) for \( t < 0 \). \( \beta = \sqrt{\nu^2 - 1} \) here.

First "rotate" the coordinates by 45° and write the equations in terms of
\[ \xi = \frac{E - Y}{\sqrt{2}}, \quad \eta = \frac{X + Y}{\sqrt{2}} \]

Then \( (A.6) \iff u_{\xi\eta} = \frac{1}{\beta} \delta(\eta) \delta(\xi) \) \( \text{(A.7)} \)
while the solution in the figure is
\[ u = \frac{1}{\beta} H(\xi) H(\eta) \]  \( \text{(A.8)} \)
where \( H \) is the Heaviside function
\[ H(\xi) = 0, \quad \xi < 0 \quad \text{and} \quad H(\xi) = 1, \quad \xi > 0. \]

Clearly \( (A.8) \) satisfies \( (A.7) \), and vanishes for \( t < 0 \).