Point source for the wave equation

(Constant speed) [Elliptic/hyperbolic transition]

Problem \( u_{tt} - \Delta u = \delta(x-vt) \delta(y) \delta(z) \) \( \Box \)

(in general \( \delta(x_1-vt) \delta(x_2) \cdots \delta(x_d) \)

where \( d = \text{dimension} \) and \( \Delta = \sum_{n} \partial^2_{x_n} \)

Furthermore, assume \( 0 \leq v < 1 \)

Is there a steady state solution?

Try \( u = u(x, y, z), x = \frac{x-vt}{\sqrt{1-v^2}} \) \( \Box \)

where the factor \( \frac{1}{\sqrt{1-v^2}} \) is a useful normalization \( \Box \)

Lorentz contraction

Then (1) reduces to \( \Delta u = -\delta(\beta x) \delta(y) \delta(z) \) \( \Box \)

\( \beta = \sqrt{1-v^2} \)

and \( \Delta = \partial^2_x + \partial^2_y + \partial^2_z \)

\( \Box \)

† It makes the coeff. of \( \partial^2_x \) in (3) one.

†† We now use the fact that \( \int \beta \delta(\beta x) \phi(x) \, dx = \frac{\phi(y)}{\beta} \, dy = \phi(0) \).
Write
\[ \Delta u = -\frac{1}{\beta} \delta(x) \delta(y) \delta(z) \]
4

In addition we want \( u \) to decay at \( \infty \).
\[ \Rightarrow \quad u = \frac{1}{4\pi \beta \rho} \quad (\text{no signals from } \infty) \]
5

See appendix 1

Note that \( \Phi = \rho \) solves
\[ \Phi^2 - (\nabla \Phi)^2 = 0, \]
the characteristic equation.
6

Since \( \Phi_y = \frac{x}{\rho}, \Phi_x = \frac{y}{\rho}, \Phi_t = -\frac{\nu}{\beta} \frac{y}{\rho}, \Phi_x = \frac{1}{\beta} \frac{y}{\rho} \).

These wave fronts are ellipsoidal with axis in the direction of propagation shrunk by a factor \( \beta \).

Of course, (5) is valid only if the source has been active for all time.

If, say, \( u = 0 \) at \( t = 0 \); then we cannot use (5) far from the source.

Note: beware of using these solutions for, say,
sound unless $V < 1$. The reason is that in these systems the wave equation is valid only for small perturbations. The original systems are Galilean invariant and the wave equation is not, so the solutions diverge drastically as $V$ approaches 1.

**What happens in 2-D?** Well then the fundamental solution does not decay at $\infty$ and we cannot use it. Reason is that there is no steady state 1-D example. $u_t - u_{xx} = \delta(x-Vt)$, $0 < V < 1$, and $u = u_t = 0$ at $t = 0$.

Then the solution ahead of the $\delta$ should be a right-going wave, and the solution behind a left-going wave. Thus

$$u = f \left( \frac{t-x}{1-V} \right) \text{ for } x > Vt$$

$$u = g \left( \frac{t+x}{1+V} \right) \text{ for } x < Vt$$
with the conditions:

\[ u \text{ continuous at } x=vt \implies f(t) = g(t) \]  

[reason for the scallop of the argument of \( f \) and \( g \) is to make them equal at \( x=vt \)]

\[ -v [u_x] = [u_x] = 1 \]  

where \[ [p] = p_{\text{ahead}} - p_{\text{behind}} \]

This last one yields

\[ 1 = -v \left\{ \frac{1}{1-v} f'(t) - \frac{1}{1+v} g'(t) \right\} - \left\{ \frac{-1}{1-v} f'(t) - \frac{1}{1+v} g'(t) \right\} \]

\[ = f'(t) + g'(t) \]  

(8a) and (8b) \implies

\[ f(x) = g(x) = \frac{1}{2} x + \text{const.} \]

However, we also want \( u = 0 \) for \( x > t \) on \( \mathbb{Z} \)

\( x < t \) (causality).

\[ \implies \text{const} = 0 \]

\[ u = \frac{1}{2(1-v)} (t-x) \text{ for } vt < x < t \]

\[ u = \frac{1}{2(1+v)} (t+x) \text{ for } -t < x < vt \]

0 elsewhere
Note that in the limit \( v \to 1 \) the solution develops a discontinuity at \( x = t \)

\[
u = \frac{1}{4} (t + x) \quad \text{for} \quad -t < x < t
\]

\( u = 0 \) elsewhere.

What happens if \( v > 1 \)

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 8(x - vt)
\]

\( v > 1 \)

\[
 u = 0 \quad \text{at} \quad t = 0
\]

\[
 u_t = 0
\]
Thus
\[ u = f\left( \frac{x-t}{\sqrt{1-v^2}} \right) - g\left( \frac{x+t}{\sqrt{1-v^2}} \right) \quad \text{for } x \leq vt \]
and
\[
\begin{align*}
    f(\xi) &= 0 \quad \text{for } \xi \leq 0 \\
    g(\xi) &= 0 \quad \text{for } \xi \leq 0 \quad \text{I. Cond. (14)}
\end{align*}
\]
\[ f(\xi) = g(\xi) \quad \text{for } \xi > 0 \quad (u=0 \text{ at } x=vt) \]

And at \( x=vt \)
\[
1 = -v \left[ \frac{u_t}{u_x} \right] - \left[ u_x \right] = \\
= v \left[ \frac{-1}{\sqrt{1-v^2}} \frac{f'}{\sqrt{1-v^2}} - \frac{1}{\sqrt{1-v^2}} \frac{g'}{\sqrt{1-v^2}} \right] + \left[ -\frac{1}{\sqrt{1-v^2}} \frac{f'}{\sqrt{1-v^2}} - \frac{1}{\sqrt{1-v^2}} \frac{g'}{\sqrt{1-v^2}} \right] \\
= -f' - g' \\
\Rightarrow \quad f' = g' = -\frac{1}{2} \xi \quad \text{for } \xi \geq 0 \quad (15)
\]

\underline{Note:} the limit \( v \downarrow 0 \) gives the same answer as in (12).

\underline{Note:} the initial value problem in 2 and 3D can also be done "exactly" using the Green's functions for the wave equation and Duhamel's principle, but the calculations are a bit hellish.
Let us now go back to (1) and consider the case \( v > 1 \) and \( d = 2 \). Then write

\[
U = u(\tau, y) \quad \text{where} \quad \tau = \frac{vt - x}{\sqrt{v^2 - 1}}
\]

\[
U_{\tau\tau} - U_{yy} = S(-\sqrt{v^2-1}) S(y) = \frac{1}{\sqrt{v^2-1}} S(\tau) S(x)
\]

We also expect that, "head" of the \( S \), that is \( x > vt \), it should be \( U = 0 \)

\[
\Rightarrow \tau < 0 \quad \text{i.e.} \quad \tau < 0
\]

We end up with an initial value problem for the wave equation \( U_{\tau\tau} - U_{yy} = 0 \) where at \( \tau = 0 \) \( U = 0 \) and \( U_\tau \) jumps from 0 to \( 1/\sqrt{v^2-1} \) at \( y = 0 \) only and it is zero elsewhere.

**Solution**

\[ u = 0 \quad \text{at} \quad \tau = 0 \quad \text{and} \quad y = \pm \sqrt{v^2-1} \]

\[ \tau \]

\[ u = \frac{1}{\sqrt{v^2-1}} \quad \text{at} \quad y = 0 \quad \text{and} \quad u = 0 \quad \text{elsewhere} \]
Again, you can check that \( \phi = \sqrt{y^2 + z^2} \) solves the characteristic equation \( \phi_t - (\nabla \phi)^2 = 0 \).

Note that this solution is \textit{unusual} (discontinuous) at \( y = \pm T \). The solution in 3-D is harder to obtain (but it also reduces to an i.v. problem) and it is also unusual (though not discontinuous) at \( \sqrt{y^2 + z^2} = T^2 \).

Mach cone

Cherenkov radiation

\[
\text{fast particle} \quad c_m < \sqrt{c_0}
\]

In nonlinear problems a shock forms and replaces the Mach cone: Sonic booms

However, remember: sonic booms, Galilean invariant

opus; Cherenkov (Lorentz invariant) similar / Not same!
Appendix 1. Fundamental solution of the
Laplace equation.

where $\Delta = \sum \delta_{x_i}$.

We claim that solutions are:

1. $\Phi = -\frac{1}{2\pi} \ln(r)$ in 2-D \(a.2\)

2. $\Phi = \frac{1}{4\pi r}$ in 3-D and, generally \(a.3\)

3. $\Phi = \frac{1}{(d-2)\alpha_d r^{d-2}}$ for $d \geq 3$ \(a.4\)

where $r = \sqrt{\sum x_i^2}$ and $\alpha_d$ = "area" of unit sphere.

Note: Any other solution of (a.1) which is bounded at $\infty$ will differ from $\Phi$ above by a constant.

The reason is Liouville's Theorem, that says that any bounded harmonic function is constant.

The case $d=2$ is a bit different because then (a.1) has no solutions bounded at $\infty$. We then require $\nabla \Phi$ to vanish at $\infty$, instead of bounded at $\infty$. 
Proof. For functions of \( r \) only, \( \Delta \Phi = r^{1-d} (r^{d-1} \Phi )' \).

It is then easy to show that \( \Phi \) satisfies \( \Delta \Phi = 0 \) for \( r \neq 0 \). Thus we only need to check the behavior at \( r=0 \). Now, for any test function \( \Phi \), by definition it must be \( \dagger \)

\[
\Phi (0) = - \int (\Delta \Phi ) \Phi = - \int \Phi \Delta \Phi = - \frac{r^{d-1}}{d-1}
\]

\[
= - \frac{d}{d-1} \int _0 ^\infty \Phi r^{1-d} (r^{d-1} \Phi ') dr = - \frac{d}{d-1} \int _0 ^\infty (r^{d-1} \Phi ') dr
\]

\[
= - \frac{d}{d-2} \left[ \Phi \right] _0 ^\infty - \int _0 ^\infty \Phi ' dr = \Phi (0) \checkmark
\]

The calculation for \( d=2 \) is a bit different, but works the same way.

\( \dagger \) Here I assume \( \Phi = \Phi (r) \) for simplicity. But, if not, note \( \int (\Delta \Phi ) \Phi = \int (\Delta \Phi ) \bar{\Phi} \), where \( \bar{\Phi} = \bar{\Phi} (r) \) is the average of \( \Phi \) over the angular variables.