[T1] Breaking/no breaking; Genuine nonlinearity and linear dependency

23 A characteristic field that is genuinely nonlinear, \((\tilde{R}_m \cdot \nabla \tilde{\zeta}) \lambda_m \neq 0\), can lead to characteristics that "cross" and "wave-breaking" (derivatives cease to exist, develop \(\infty\)). However, this is not always true! It is not so even for the scalar case, it depends on the initial and boundary conditions. In the scalar case it is easy to characterize which initial conditions lead to wave breaking, and which do not. I do not know if such characterization is possible, or known, for systems. See [T3] p23.17, [T9] & [T5] p 23.20

24 Linearly degenerate characteristic fields cannot "break" on their own. That is: the

\[ (R_m \cdot \nabla \zeta) \lambda_m = 0 \]
simple wave solutions in (2-3) have \( \lambda \) constant. On the other hand, singular behavior in them can be triggered by the other characteristics. For example, in Gas Dynamics, when an acoustic simple wave, (2-3), breaks and triggers a shock, the solution stops being a simple wave and singular behavior in the other fields may appear. For that matter, when shocks collide, they can generate discontinuities in the particle path (linearly degenerate) characteristic; contact discontinuities.

However, in Gas Dynamics (and other physical examples) the linearly degenerate characteristic fields are associated with the particle paths, which cannot cross. Thus the question arises: is 

\[ \text{This is a physical argument. I do not know how to math-prove it.} \]

\[ \text{For weakly nonlinear situations these effects are higher order, and can be ignored with "small" error.} \]
this true in general, for all linearly degenerate characteristic fields? I do not know the answer.

Note that this is not a simple question; of course the characteristics do not cross as long as the slms. remain smooth. The question is what happens afterward, and this requires a theory "beyond" breaking, which (for the general case) largely does not exist.

[26] Weakly non-linear simple waves. Beyond wave breaking. Here we consider small amplitude perturbations to the constant solutions to (1); W.L.O.G. say $\bar{y} = 0$. Then, to leading order, the solution will satisfy the linearized equations. Thus it will have the form $\bar{y} \approx \varepsilon \sum_m (x - \lambda_m t) \bar{R}_m$. Then we now look for a solution that incorporates the leading order nonlinear corrections to

\[ i.e.: \text{equilibrium.} \]
the simple component linear solution $\epsilon \sigma (x - \lambda_p t) \hat{R}_p$ (some $m$). Note: here $0 < \epsilon << 1$ is a measure of the size of the perturbation. Because the solution we seek is small amplitude, the nonlinear effects are small and "build up" slowly (over time scale $\gamma \epsilon$)

Thus we look for a solution of the form

$$\vec{Y} = \epsilon \sigma (x - \lambda_p t, \tau) \vec{R}_p + \epsilon^2 \vec{Y}_2 (x, \tau) + ...$$

$$x \quad \tau = \epsilon t$$

(27)

For simplicity we will also assume periodic behavior in $x$.

**Notation:** We assume that the $\vec{Y}$ are conserved quantities and that (1) has a conservation form $\vec{Y}_t + F(\vec{Y})_x = 0$ so that $A$ is the gradient of $F$. Furthermore, expand $F$:

$$F(\vec{Y}) = A_0 \vec{Y} + Q_0 (\vec{Y}, \vec{Y}) + ...$$

(28)

where $A_0$ is a square matrix with eigenvalues $\lambda_m$ and eigenvectors $\hat{\lambda}_m, \vec{R}_m$; and $Q_0$ is
$\Omega_0$ is a vector-valued symmetric bilinear form.

Then we substitute (27) into $\tilde{Y}_t + \tilde{F}(\tilde{Y})_x = 0$, and collect equal powers of $E$. The order $E$ terms yield the linear eqn., satisfied by (27). At $O(E^2)$ we find

$$(A_0 - \Lambda_0^0 I) \tilde{Y}_2 \chi = -\left\{ \sigma_{1} R_{p} + (\sigma^2)_{k} \right\}_{0} (\tilde{R}_{p}, \tilde{R}_{p})$$

For this to have a solution the r.h.s. must be orthogonal to $L_0^0$, that is

$$\sigma_{1} + (\frac{1}{2} \chi \sigma^2) \chi = 0,$$

where $\chi = 2 \, L_0^0 \cdot \Phi(\tilde{R}_{p}, \tilde{R}_{p})$. We also need the r.h.s. in (29) to have mean zero, but (30) guarantees this. So, given (30), (29) can be solved to find a periodic in $\chi \tilde{Y}_2$.

\textbf{Note 1} since $\tilde{Y}$ is conserved, $\sigma$ is conserved (up to the order considered). Thus (30) is a construction form that we can use to introduce shocks into $\tilde{Y}$.

\textbf{\dagger} Write $\tilde{Y}_2 = \sum \mu_m \tilde{R}_{m}$ and solve for the $\mu_m$. 

Note 2: It can be shown that shocks in $\Omega$, at the level of (30), do not break down the approx. in (27). That is, we do not need to introduce any components in the solution, at $O(\varepsilon)$, which involve $\vec{R}_m$ for $m \neq p$. The shocks trigger such components, but at higher orders [$O(\varepsilon^2)$ and beyond].

Note 3: $\lambda^+$ in (30) in the Genuine Nonlinearity coefficient at $Y = O(\text{linearization state})$

Proof: The Genuine nonlinearity coefficient in question is $\lambda^+ = \left[ (\vec{R}_p \cdot \vec{V}_Y) \lambda_p \right]_{\vec{V}_Y = 0}$.

Let now consider $\lambda_p = \lambda_p (\varepsilon \vec{R}_p^\circ)$ then $\lambda^+ = \lambda_p \bigg|_{\varepsilon = 0}$ where $^e \frac{d}{d\varepsilon}$ in this argument.

But $\lambda_p = \vec{L}_p \vec{A} \vec{R}_p$ (all evaluated at $\vec{Y} = \varepsilon \vec{R}_p^\circ$)

$\Rightarrow \lambda_p = \vec{L}_p \vec{B} \vec{R}_p + \vec{L}_p \vec{A} \vec{R}_p + \vec{L}_p \vec{A} \vec{R}_p$

$= \lambda_p \left( \vec{L}_p \vec{R}_p + \vec{L}_p \vec{R}_p \right) + \vec{L}_p \vec{A} \vec{R}_p$

$= \lambda_p \Omega$ because $\vec{L}_p \cdot \vec{R}_p = 1$.

† Painful, but straightforward calculation
Thus \( \mathbf{g} = \mathbf{\mathcal{L}}_p \mathbf{A} | \mathbf{\dot{R}}_p \mathbf{=} 0 \). \[[A]\]

On the other hand, from (28) it is easy to see (evaluate at \( \mathbf{\dot{Y}} = \mathbf{\mathcal{R}}_p + \varepsilon \mathbf{\dot{Y}} \), \( \varepsilon \) infinitesimal) that

\[
A(\mathbf{\mathcal{R}}_p) \mathbf{\dot{Y}} = A_0 \mathbf{\dot{Y}} + 2\varepsilon Q_0(\mathbf{\mathcal{R}}_p, \mathbf{\dot{Y}}) + O(\varepsilon^2)
\]

Hence \( A | \mathbf{\dot{Y}} = 2Q_0(\mathbf{\mathcal{R}}_p, \mathbf{\dot{Y}}) \) \([2] \Rightarrow \mathbf{g} = \mathbf{\dot{Y}}\)

\( \varepsilon \) And keep the terms linear in \( \mathbf{\dot{Y}} \); by definition, this is \( A(\mathbf{\mathcal{R}}_p) \)

\[[T3]\] An argument for breakdown in finite time

Consider a system of the form in (1) that can be written in terms of Riemann Invariants

\[ \eta = 1 : N \quad \Gamma_\eta = \text{constant along } \frac{d}{dt} \mathbf{x} = \lambda_\eta(\mathbf{\Phi}) \]

and assume initial conditions such that \( \Gamma_\eta = 0 \) \( \eta = 1 : N \) outside some finite interval \( \alpha \leq x \leq \beta \)

\( \varepsilon \) This is always true if \( N = 2 \).
Furthermore, assume that there exist constants \( \{\tilde{u}_j\}_{j=1}^{N-1} \) and \( \{\tilde{d}_j\}_{j=2}^N \) such that

\[
\lambda_1 < \tilde{u}_1 < \tilde{d}_2 < \lambda_2 < \tilde{u}_2 < \tilde{d}_3 \ldots \tag{33}
\]

\[
\ldots \lambda_{N-1} < \tilde{u}_{N-1} < \tilde{d}_N < \lambda_N
\]

for all the possible values \( p \) takes for the initial conditions.

**Example**: Isentropic Gas Dynamics, where \( \lambda_1 = u - a \) and \( \lambda_2 = u + a \) (a sound speed).

If the initial conditions are "not too far" from a constant state, then it will be \( a > a_0 \) and \( |u - u_0| \leq M \), where \( 0 < M < a_0 \). Then:

\[
u - a = (u - u_0) - a + u_0 \leq M - a_0 + u_0 = \tilde{u}_1
\]

\[
u + a = (u - u_0) + a + u_0 \geq a_0 - M + u_0 = \tilde{d}_2
\]

and \( \tilde{u}_1 < \tilde{d}_2 \)

In this case we will show that, if some of the characteristics are genuinely nonlinear,
breakdown of the solution in a finite time is almost certain.

For suppose that the solution is differentiable for all times.

Then let $\alpha_n$, $n=1:N$ be the $n^{th}$ characteristic starting at $\alpha$, and similarly for $\beta_n$. Then $\Gamma_n \equiv 0$ outside the interval $\alpha_n \leq x \leq \beta_n$. Further, because of (33) these intervals become disjoint in a finite time, at which point the solution splits into $N$ simple waves.

Then if any of the genuinely nonlinear fields have a non-trivial $\Gamma_n$, breakdown occurs in a finite time (see (35)).

Why? Because $\Gamma_n$ has to return to zero at $\alpha_n$ and $\beta_n$. Thus, if it is nontrivial between
\( \alpha_n \) and \( \beta_n \) at a place where \( \lambda_n \) is decreasing will exist—precisely the condition for 
\[(\Gamma_n)_t + \lambda_n (\Gamma_n)_x = 0 \]
to breakdown in a finite time.

\[ \text{Note: Because } \Gamma_n \text{ is constant along the } \eta^{th} \text{ characteristic, a nontrivial } \Gamma_n \text{ after the separation into simple waves is guaranteed if } \Gamma_n \text{ is nontrivial at time } t=0. \quad (35) \]

\[ \text{[T4]} \]
When a Riemann invariant form is not available, the arguments above do not apply, but analysis is still possible. \[ \text{See:} \]

Formation of singularities in one-dimensional nonlinear wave propagation.


\[ \text{[T5]} \]

Note that the arguments above fail when the solutions do not vanish outside some interval; for example, they are periodic.
In this case there is evidence that nontrivial solutions (with ALL the wave-modes excited and non-trivial) exist that DO NOT ever break.

I will add references to this stuff with an update to this lecture later.