Simple Waves (Generic Case; no Riemann variables)

Consider the system

$$\dot{\vec{Y}} + A \vec{Y}_x = 0 \tag{1}$$

where $A = A(\vec{Y})$ is an $N \times N$ matrix, and $\vec{Y}$ is an $N$-column-vector. Assume now that $A$ has a real eigenvalue $\lambda = \lambda(\vec{Y})$ with eigenvector $\vec{R} = R(\vec{Y})$.

We will also assume that $\vec{R}$ depends "smoothly" on $\vec{Y}$. Let now $\vec{U} = U(\vec{Y})$ be a solution to the ode

$$\frac{d\vec{U}}{d\vec{Y}} = \vec{R}(\vec{U}). \tag{2}$$

We now claim that $\vec{Y} = U(\varphi)$, $\varphi = \varphi(x,t)$ solves (1) iff.

$$\varphi_t + \lambda \varphi_x = 0. \tag{3}$$

Proof

$$\lambda = \lambda(\varphi) = \lambda(U(\varphi))$$

Substituting $\vec{Y} = U(\varphi)$ into the equation, and using (2) yields

$$(\varphi_t + \lambda \varphi_x) \vec{R} = 0.$$ This proves the

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# All we need to guarantee solutions is $\vec{R}$ Lipschitz!

† In a strictly hyperbolic system this follows if $A$ depends "smoothly" on $\vec{Y}$. We also assume $\lambda$ depends "smoothly".
result because $\tilde{B} \neq 0$.

Solutions of the type in (3) are called \underline{simple waves}. Note that the example in pp. 22.04-22.05 are a special case of this. When $N$ Riemann variables exist, changing variables to the $\{\Gamma_n\}$ diagonalizes $A = \text{diag}(\lambda_n)$. Then take

$$\tilde{U} = (\tilde{\xi}, \tilde{a}_2, \ldots, \tilde{a}_N)^T$$

in (2). This yields the soln. in (14-16) pp. 22.04-22.05.

\begin{itemize}
  \item \underline{Wave Breaking}: Equation (3) develops wave breaking $\{\text{derivatives blow up}\}$ and wave steepening $\frac{d\lambda}{d\varphi} \neq 0$.
\end{itemize}

This shows that the solutions to (1) cannot, in general, be assumed to exist for all times—even if the initial data are smooth. In general, beyond some critical time $t_c$, the solutions

\[\text{This is (13) p. 22.03}\]
close to exist as classical solutions and something must be done to continue them beyond $t_c$ [e.g.: shocks for gasdynamics, etc.; caustic expansions for geometrical optics; and dispersive wave modulation; etc.]

**Genuine nonlinearity.** Since $\lambda = \lambda(\nabla(\varphi))$ and $\frac{d\varphi}{d\varphi} = \tilde{R}$, the condition for wave-breaking in (5) can be recast in the form

$$\tilde{R} \cdot \nabla_{\varphi} \lambda \neq 0,$$

where $\nabla_{\varphi} = (\partial_{\varphi_1}, \ldots, \partial_{\varphi_n})^T$. A characteristic field that satisfies (6) is called **Genuinely Nonlinear**.

By contrast, a characteristic field for which

$$\tilde{R} \cdot \nabla_{\varphi} \lambda \equiv 0$$

is called **Linearly Degenerate**.

**Example** Gas Dynamics. Write the equation in the form

$$p_t + (p u)_x = 0, \quad u_t + u u_x + \frac{1}{\rho} p_x = 0, \quad s_t + u s_x = 0.$$
where \( p = p(p, s) \) [This form is valid only when the solution has derivatives]. Thus corresponds to
\[
\tilde{\mathbf{Y}} = (p, u, s)^T \quad \text{and} \quad \mathbf{A} = \begin{bmatrix}
\frac{\partial}{\partial p} & 0 & 0 \\
\frac{\partial}{\partial u} & \frac{\partial}{\partial u} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
where \( \frac{\partial^2}{\partial p} \), \( \sigma = \frac{\partial^2}{\partial s} \).

Then \( \lambda = u + a \); \( \mathbf{R} = (p, a, 0)^T \); and:
\[
(\mathbf{R}, \nabla \tilde{\mathbf{Y}}) \lambda = \frac{\partial}{\partial p} \lambda + \frac{\partial}{\partial u} \lambda = \frac{\partial}{\partial p} (pa)
\]

Generally, for gases: \( \frac{\partial}{\partial p} (pa) > 0 \)

\[ \text{Genuine nonlinearity} \]

For polytropic gases \( p = K(s) \rho^{\gamma}, \gamma > 1 \):
\[ a = \sqrt{\gamma K} \rho^{\gamma-1}/2; \quad pa = \sqrt{\gamma K} \rho^{\gamma+1}/2. \]

On the other hand \( \lambda = u \); \( \mathbf{R} = (\sigma, 0, a)^T \)

so that \( (\mathbf{R}, \nabla \tilde{\mathbf{Y}}) \lambda = \sigma \frac{\partial}{\partial p} \lambda + a^2 \frac{\partial}{\partial s} \lambda = 0 \)

\[ \text{Linear dependency.} \]

\[ \text{Note: For a typical gas, write } p = p(v, s), v = 1/p \]
Then \( \frac{\partial}{\partial v} < 0 \) and \( \frac{\partial^2}{\partial v^2} p > 0 \)
Since \( \frac{\partial p}{\partial p} = -V^2 \frac{\partial^2 p}{\partial V^2} \) , \( \rho^2 a^2 = \rho \frac{\partial}{\partial p} p = -\frac{\partial p}{\partial V} \) and

\( p a \) increases with \( p \) iff \( (p a)^2 \) increases with \( p \)

iff \( \frac{\partial}{\partial V} \) decreases with \( p \) iff \( \frac{\partial}{\partial V} \) increases with \( V \)

**Physically**: As a gas expands adiabatically, the pressure decreases, and the rate of decrease decreases as well \([\text{as } V \to \infty \text{ the pressure should go to zero!}]\)

**Domains of dependence and influence**

We are going to show now that, for a hyperbolic system, no information propagates faster than the fastest characteristic or slower than the slower characteristic.

Thus, for example, means that if the initial cond.

\[ \text{And a strong solution (derivatives exist). When shocks appear, then the entropy guarantee that this remains true.} \]
are modified in some interval $a < x < b$, then the solution only changes in the region $X_a \leq x \leq X_b$ where $X_a(t)$ is the slowest characteristic starting at $a$, and $X_b$ is the fastest characteristic starting at $b$. Similar statements can be made about changes to the boundary data; as to ascertaining which part of the data affects the solution at some point in space-time.

Domain of dependence.

Note: Ch. need not be straight! \{ This is just a bad drawing. \}

Warning: the argument is intuitive, not rigorous!
We now write the system in characteristic form, using (13-44) in p. 17.15, where the \( \hat{L}_m \) are N linearly independent vectors \( \hat{L}_m \) and \( \hat{L}_m = \hat{L}_m(\vec{Y}, x, t) \), \( \lambda_m = \lambda_m(\vec{Y}, x, t) \), etc.

Next we compute what happens in one "infinitesimal" time step, \( \vec{Y}(x, t) \rightarrow \vec{Y}(x, t + dt) \), using (15). Let \[ \hat{L}_m(x, t + dt) = \hat{L}_m(x, t) + \lambda_m dt \] below apply, with:

\[
L_m \cdot [\vec{Y}(x, t + dt) - \vec{Y}(x, t)] = f_m dt
\]

Thus

\[
\vec{Y}(x, t + dt) = \sum_m \left[ L_m \cdot \vec{Y}(x, t) + f_m dt \right] \hat{R}_m.
\]

If we also need the \( \hat{R}_m \)'s, with \( \hat{L}_p \cdot \hat{R}_q = \delta_{pq} \) (left and right eigenvectors \( A \))

\[
\uparrow \quad \text{Evaluate } \hat{L}_m, \hat{R}_m, \lambda_m, \text{ and } f_m \text{ at } (\vec{Y}(x, t), x, t) \text{ in (16-17) above.}
\]
(17) expresses the solution at \((x, t + dt)\) in terms of the values of \(\vec{Y}\) at "the feet" of the characteristics through \((x, t + dt)\). That is, the values \(\vec{Y}(x_m, t)\). Extend now this to many "time steps" we see that the solution at \((x_0, t_0)\) depends only on the values in the region \(x_b(t) < x < x_2(t)\) \(t \leq t_0\) and \(x_1, x_2\) fastest and slowest through \((x_0, t_0)\).

**Note:** (17) is actually a "numerical" method for computing the solution (when \(dt\) is small but...
not infinitesimal). To make the argument rigorous "all" we need to do is to prove that (17) converges to the solution as \( \Delta t \to 0 \), provided that the solution is smooth enough. For this we assume that the solution exists, and go from there. This is relatively easy. Proving that (17) yields a solution for \( 0 < t < T \) (some \( T \)), provided the data is smooth enough is harder.

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Shocks. What happens when the solutions no longer have derivatives and, say, shocks are introduced?

I will do this only with a couple of examples because, in fact, there does not exist a theory for "general" systems of the form in (1). Even for specific examples such as geodynamics there are still open questions. Explain
Example:

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0 \]

\[ u(x,0) = \begin{cases} 2 & x < 0 \\ 0 & x > 0 \end{cases} \]

The shock introduces a speed of propagation that is not a characteristic. Yet it is still between the max. and min. of the char. speed at \( t = 0 \).

Example: \( N = 2 \) system (sketch)

Domains of dependence in shadowed region

\[ \lambda_1 < \lambda_2 \]

The shock shown is in the \( \lambda_2 \) charact.

In 2-D the domains of dependence (or influence) become "cones" instead of triangles; "hypercone," in more than 2-D.