From Lecture 19: recap solution to linear, constant coeff., homogeneous system and →
Do boundary conditions (bottom p19.02) and forward. Then continue below:

Example: Acoustics in 1-D

\[ \begin{align*}
\text{Isentropic Euler equations} & \quad \rho_t + (\rho u)_x = 0 \\
\text{Sln. (equilibrium)} & \quad u_t + uu_x + \frac{1}{\rho} \rho_x = 0
\end{align*} \]

\[ \rho = \rho_0, \quad u = u_0 \quad \text{constants} \]

Linearize \( \rho = \rho_0 + \delta \rho, \quad u = u_0 + \delta u \)

\[ \begin{align*}
(\delta \rho)_t + u_0(\delta \rho)_x + \rho_0(\delta u)_x = 0 \\
(\delta u)_t + \frac{a_0^2}{\rho_0}(\delta \rho)_x + u_0(\delta u)_x = 0
\end{align*} \]

\[ \lambda_{\pm} = u_0 \pm a_0 \]

\[ \begin{align*}
R_{\pm} & = \begin{bmatrix} \rho_0 \\ \pm a_0 \end{bmatrix}, \quad L_{\pm} = \frac{1}{2 \rho_0 a_0} \begin{bmatrix} a_0, \pm \rho_0 \end{bmatrix}
\end{align*} \]

Check \[ \begin{align*}
AR_{\pm} & = \lambda_{\pm} R_{\pm}, \quad L_{\pm} A = \lambda_{\pm} L_{\pm}
\end{align*} \]
and \( L_+ R_- = L_- R_+ = 0,\) \( L_+ R_+ = L_- R_- = 1 \)

**Generalized Hyperbolic system**

If in a system of conservation laws the independent variables are not the conserved quantities, then system:
\[
\begin{align*}
\dot{\mathbf{u}}(\mathbf{u})_t + \mathbf{F}(\mathbf{u})_x = 0 \\
\text{expanded as more}\end{align*}
\]

\[
\mathbf{B} \mathbf{u}_t + \mathbf{A} \mathbf{u}_x = 0
\]

We expect \( \mathbf{B} \) to be invertible, and in this case (1) can be written in standard form
\[
\mathbf{\tilde{u}}_t + \mathbf{B}^{-1} \mathbf{A} \mathbf{\tilde{u}}_x = 0
\]

However, *sometimes* (1) is more convenient.

Similarly, for a system in more than 1-D e.g.
\[
\mathbf{\tilde{u}}_t + \mathbf{A} \mathbf{\tilde{u}}_x + \mathbf{B} \mathbf{\tilde{u}}_y = 0
\]

one may want to look at steady states
\[
\mathbf{B} \mathbf{\tilde{u}}_y + \mathbf{A} \mathbf{\tilde{u}}_x = 0
\]

and in this case there is no particular reason to write something like (4).

† Here \( \mathbf{A} \) and \( \mathbf{B} \) are square matrices
So let us consider systems of the form
\[ B \vec{u}_y + A \vec{u}_x = \vec{F} \quad \text{(6)} \]
\[ B = B(\vec{u}_x, y), \quad A = A(\vec{u}_x, y), \quad \vec{F} = \vec{F}(\vec{u}_x, y) \]

The condition for weak linearization is now
\[ (\Phi_y B + \Phi_x A)[u_\phi] = 0 \quad \text{(7)} \]

which is now a generalized eigenvalue problem.

We now make the non-degeneracy assumption

There is one \( \Theta \), at least, such that
\[ \det \left[ (\sin \Theta) B + (\cos \Theta) A \right] \neq 0 \quad \text{(8)} \]

Why?

(1) Note that for (3) \( B \) invertible means that
\[ \Theta = \pi/2 \] satisfies (8)

(2) If (8) fails all directions are characteristic.

An equation with this property means that the system is under-determined.

Example: \[ u_y + u_x = 0 \]
\[ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]

More on this later.
The condition in (8) means that
\[ \text{det}(\alpha B + \beta A) = \text{Tr}(\mu_n \alpha + \nu_n \beta) \]
\[ N \times N \text{ matrix} \quad N \text{-factor} \]
where \( |\mu_n|^2 + |\nu_n|^2 = 0 \) (factorization) is full.

Then (6) is **hyperbolic** if the problem
\[ [(\sin \theta_n) B + (\cos \theta_n) A] R_n = 0 \]
has \( N \)-linearly independent eigenvectors, and **strictly hyperbolic** if all the eigendirections \( \theta_n \) are distinct.

This then means that (6) can be written as an equivalent system of \( N \) equations, each involving derivatives in the plane along a single direction (given by \( \theta_n \)).

So, again "\( N \)" waves, even though there is no time.
Examples of what can happen when the hyperbolic condition fails.

**Example** \( \dot{\vec{Y}}_t + A \vec{Y}_x = 0 \) \( \text{(12)} \)

A \( nxn \) matrix, but eigenvalues not all real and \( C \) real

Then, for some \( R \), \( AR = \lambda R \) \( \lambda \) * conjugate and \( AR^* = \lambda^* R^* \)

with \( \text{im}(\lambda) \neq 0 \) and \( R \neq 0 \)

Let \( \lambda = a + ib \), \( a \) and \( b \neq 0 \) real

The (12) has solutions

\[ Y = e^{ikx + \mu t} R \] \( \text{(13)} \)

where \( \mu = -ik\lambda = -ika + bk \)

and \[ Y = e^{ikx + \mu t} R^* \] (\( k \) real)

with \( \mu = -ik\lambda^* = -ika - bk \)

Note that these solutions have arbitrarily large exponential growth for short frequencies \( |k| \gg 1 \).

Problem is ill posed as an evolution in time \( \text{(14)} \).
Note: for steady state solutions [e.g. (5)] this need not be a problem. It just means that the steady state cannot be specified by conditions on "one side" only (e.g.: in a flow with flow velocity less than sound, the waves can propagate up and down the flow).

Example Algebrac multiplicity > geometric

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
u_t + uu_x &= 0
\end{align*}
\]
\[A = \begin{bmatrix} u & \rho \\ 0 & u \end{bmatrix}\]

1 real eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1.

Then \( \frac{du}{dt} = 0 \) along \( \frac{dx}{dt} = u \)

So \( u \) breaks and produces shocks or something. But what about \( \rho \)?

Let \( \frac{da}{dt} = u \) and \( \frac{db}{dt} = u \) charact. (17)
Then
\[ \frac{d}{dt} \int_{a}^{b} p \, dx = b p_b - a p_a + \int_{a}^{b} p_t \, dx \quad (18) \]
\[ = b u_b p_b - a u_a p_a - \int_{a}^{b} (pu)_x \, dx = 0 \]

That is: the "amount" of \( p \) between characteristics is conserved \( \Rightarrow \) when characteristics cross, \( p \to \infty \)

\[ \text{wave: at a "shock" in } u, \text{ } p \text{ develops a } \delta \text{ function. But then, what is the meaning of } pu? \]

This problem appears for traffic flow with no preventive driving:
\[ p_t + (pu)_x = 0 \quad (20) \]
\[ u_t + uu_x = \frac{1}{T} (U-u) \]

which makes this a very bad model for traffic flow!
Another context where a situation like the one in eqn. (15) shows up is "dispersive wave modulation theory."

A linear (homogeneous) dispersive system is one that accepts solutions with space-time dependence of the form

\[ e^{i\theta}, \quad \theta = kx - \omega t \]

(proportional to \( e^{i\theta} \))

\[ -\infty < k < \infty \]

\[ \therefore \quad \vec{u} = \vec{Y} e^{i\theta}, \quad \vec{Y} = \text{constant} \]

\[ \omega = \Omega(k) \]

\[ \Omega \text{ real valued} \]

One can then consider "slowly varying" waves, where at each point in space-time the solution has a well-defined wave number \( k \) and wave frequency \( \omega \), but \( k \) and \( \omega \) are not constant [ \( k = k(x,t) \), \( \omega = \omega(x,t) \) ] and vary over distances and periods much longer than the wavelength. When solving the pde using Fourier, the modes neither die nor decay. "Wave,"
The wavelength $\lambda = 2\pi/k$ and the wave period $T = 2\pi/\omega$. Conservation of waves then leads to the equation

$$k_t + \omega_x = 0 \quad \iff \quad k_t + c_g(k) k_x = 0$$

$$\omega = \Omega(k)$$

(At least for smooth solutions)

But to fully characterize the wave, we also need an equation for the amplitude $a = a(x,t)$ (also assumed to be slowly varying). Turns out this equation is

$$(a^2)_t + (c_g a^2)_x = 0 \quad (23)$$

**Interpretation:** Conservation of wave-energy.

The wave energy density has the form

$$E = e(k) a^2,$$

and you can/should check that $(22-23) \Rightarrow E_t + (c_g E)_x = 0 \quad (24)$

This is needed to make sense of the concept of a local wave-length and period. {Else one cannot define a $k$ or $\omega$!}
Conversely: (22) & (24) \Rightarrow 23

What (24) says is that the wave energy is conserved, and it flows at the group speed \( c_g \).

**What is \( e(k) \)?** This is a system function that determines how much energy a wave number carries. For example, imagine that in the system, the energy density is \( u_t^2 + u_x^2 \) (e.g. something like the wave equation).

Then for \( u = a \cos \theta, \theta = kx - \omega t, \)

\[
u_t^2 + u_x^2 = (\omega^2 + k^2) a^2 \cos^2 \theta
\]

Thus the average energy per wave length (This in \( E \)) is \( \frac{1}{2} (\omega^2 + k^2) a^2 \).

If the energy is given by some other expression, e.g. \( u_t^2 + \alpha u_x^2 + \beta u_{xx} \), then \( e(k) \) changes. But the form \( E = e(k) a^2 \) in generic.
Now back to (22-23), which we write as

\[ \begin{align*}
    k_t + C_g(k)k_x &= 0 \\
    p_t + (C_g(k)p)_x &= 0 \\
\end{align*} \tag{25} \]

\( p = E \). Now this system has exactly the same problem as (15)! If \( \frac{d}{dk} C_g(k) \neq 0 \), the characteristics \( \frac{dx}{dt} = C_g \)
generally cross, and when that happens \( p \) becomes \( \infty \)!

\underline{Resolution:} In this case, we have a more detailed theory underlying (28) \([i.e.: \text{there is some wave action being behind (28)}]\). When the characteristics cross, the slowly varying assumption fails, and we need to look into more detail at what happens. This can be done \([ \text{e.g. see 18.376/377} ] \) and the answer is
(i) Multiple valued solutions to (28) are fine. We do not need, nor do they arise in the physics, shocks.

(ii) Where the characteristics cross, the amplitude becomes large, but not infinity. The amplification factor is $e^{-\frac{1}{6}}$.

(iii) (28) "foils" in a region of width $e^{\frac{1}{3}}$ in space-time. Here $0 < e << 1$ is the slowness parameter.

$e = \frac{\lambda}{L}$, where

$\lambda = $ wave-length

$L = $ length over which $a, k, \omega$ change significantly.

"smeared" $S$ function

[Non-dim units:

For $x$, typical $\lambda$

for $t$, typical $T$

Since $p \sim e^{-\frac{1}{3}}$ the amount of $p$ in caustic region is finite.
Example: Eikonal equation

This is yet another example with a situation similar to the one in equation (15).

Consider a wave front that propagates normal to itself at velocity c.

We now describe the evolution of the wavefront via a function $\phi = \phi(\vec{x})$ defined as follows:

For any point $\vec{x}_1$ let $t_1$ be the time at which the wave front goes through $\vec{x}$ then

$$\phi(\vec{x}_1) = c t_1$$

Hence the wavefront at time $t$ is given by

$$\phi(\vec{x}) = t$$

Now we write an equation for $\phi$
The unit normal to the wavefront, pointing towards the direction of propagation, is

\[ \hat{n} = (\nabla \phi) / |\nabla \phi| \]

Thus, if the wavefront is at \( \vec{x} \) at time \( t \), it will be at \( \vec{x} + c \hat{n} \, dt \) at time \( t + dt \). Then:

\[ \phi(\vec{x}) = t \quad \text{and} \quad \phi(\vec{x} + c \hat{n} \, dt) = t + dt \]

\[ \Rightarrow \quad \hat{n} \cdot \nabla \phi = 1 \quad \text{i.e.} \quad c (\nabla \phi) / |\nabla \phi| \]

\[ i.e. \quad c^2 (\nabla \phi)^2 = 1 \]

\[ \text{(33)} \]

This is the **Eikonal equation**

**Note 1** In this derivation \( c \) need not be a constant. It could be \( c = c(\vec{x}) \), or even depend on some other variable.

**Note 2** \((33)\) is the same as the equation for the weak propagation of anisotropies for the wave equation:

\[ \frac{\partial U}{\partial t} - c^2 \Delta U = \text{L.O.T} \]
Then, if the singularities are characterized by \( \psi = \text{const.} \), in the usual fashion we get

\[
\psi_t^2 - c^2 (\nabla \psi)^2 = 0
\]

(35)

Then look for solutions of the form \( \psi = \phi(x) - \tau \)

This yields (33)

For simplicity we now consider the case \( c = 1 \)

(most of what we do next carries over to the general case). Then

\[
(\nabla \phi)^2 = 1
\]

(36)

This equation has

a characteristic form

Note that this \( \rightarrow \)

is an ODE for \( \dot{x} \),

\( \phi \) and \( \nabla \phi \) along

each "ray" i.e. a solution to \( \frac{d}{dt} \dot{x} = \nabla \phi \)

Note: The 1st eqn. in (37) is the definition of ray,

the second is just the equation itself. The 3rd
follows from:
\[
\frac{d}{dt} \Phi_{x_j} = \sum_e \Phi_{x_j x_e} \frac{dx_e}{dt} = \sum_e \Phi_{x_j x_e} \Phi x_e = \\
= \frac{1}{2} \frac{\partial}{\partial x_j} (\nabla \Phi)^2 = 0
\]

Geometrically, equations (37) just retell the story: the fronts propagate normal to themselves at constant velocity \( v \). However, this means that if the wave-fronts are uncured the rays eventually cross and the front folds, with \( \Phi \) becoming multiple valued.

Edge of ray crossings "Caustic"
Again, one can ask: what happens with the wave amplitude?

The answer depends on context, but for waves (e.g. wave equation) it can be shown that

$$\nabla \cdot [\nabla \Phi \, \vec{a}^2] = 0$$  \hspace{1cm} (40)

This is, again, a conservation of energy law. It says that the wave energy flows along the ray. More specifically: energy in a "ray tube" is constant.

So, when rays cross, tubes collapse and $a \to 0$. 
Resolution The notion of a "wavefront" only makes sense as long as the scales are much larger than the wavelength. When a wavefront develops scales (e.g. curvature radius) comparable with the wavelength the approximation fails.

Just as in the modulation theory example, a thin region of width $\varepsilon^{1/3}$ develops along the caustics, where the approximation fails, and the amplitude is large $\varepsilon^{-1/6}$ but not $\infty$. Here $\varepsilon = \lambda / L$, $\lambda$ = wavelength and $L$ = characteristic length for wavefront (radius of curvature, for example).

Note You can see the caustics at the bottom of a pool on a bright day, as lines of brightness. They move because the surface changes because of waves.