Example B.2, page 18.05 (The ARZ model)

[A] Finish with the lecture 18 notes, starting with

[B] Linear, homogeneous, constant coefficients system: Exact solution.

Consider the system \( \ddot{Y}_t + A \dot{Y}_x = 0 \) \( \text{(1)} \)

where \( A \) is an \( N \times N \) constant matrix; real-diagonalizable [so \( \text{(1)} \) is hyperbolic].

Note: For a generic system \( \ddot{Y}_t + A \dot{Y}_x = F \), the "high frequency" solutions "essentially" satisfy \( \text{(1)} \), so the solutions to \( \text{(1)} \) carry this information.

Let now \( \{ \vec{R}_n \}_1^N \) and \( \{ \vec{L}_n \}_1^N \) be bases of eigenvectors for \( A \), normalized by

\( \vec{L}_n \cdot \vec{R}_m = S_{nm} \) \( \text{(2)} \)

we also have \( A \vec{R}_m = \lambda_m \vec{R}_m \) and

\( \vec{L}_m A = \lambda_m \vec{L}_m \), \( [\lambda_n] \) the characteristic speeds.
As in (46) p. 176, we introduce new variables

\[ r_m = \hat{L}_m \cdot \hat{Y}; \text{ note that } \]

then \[ \hat{Y} = \sum_m r_m \hat{R}_m \]

Left multiplying ④ by \( \hat{L}_m \), and using the fact that \( \hat{L}_m \) is a constant, yields

\[ (r_m)_t + \lambda_m (r_m)_x = \frac{d}{dt} r_m = 0 \quad \text{(4)} \]

on \( \frac{d}{dt} x = \lambda_m \).

Since \( \lambda_m \) is constant, this means that \( r_m = r_m (x - \lambda_m t) \) ⑤

Thus the general solution to ④ has the form

\[ \hat{Y} = \sum_m r_m (x - \lambda_m t) \hat{R}_m \quad \text{(6)} \]

where, for example

\[ r_m(x) = \hat{L}_m \cdot \hat{Y}_0(x), \quad \text{(7)} \]

where \( \hat{Y}_0 \) are the initial conditions.

**Boundary Conditions**: Generally, for a problem on a finite domain, b.c. are...
needed. (8) indicates what type of B.C. are needed:

At a boundary classify the \( \lambda_m \) into "outgoing" (\( \lambda_m < 0 \) on the left, \( \lambda_m > 0 \) on the right) and "incoming". Then the B.C. must be exactly enough to determine the incoming \( R_m \) in terms of the outgoing \( R_m \) (and possibly some boundary "forcing").

**Note:** If \( \lambda_m = 0 \) then \( R_m \) is determined by the i.c. only.

**Example:** (This is not constant coeff., nor homogeneous, but the moral applies)

Consider the equations

\[
 u_t + u_x = a_{11} u + a_{12} v \\
 v_t + x v_x = a_{21} u + a_{22} v
\]

for \(-1 < x < 1\).
The $\lambda = 1$ is incoming at $x = -1$ and outgoing at $x = 1$, while $\lambda = x$ is outgoing at both ends. Thus (8) requires only one bc (at $x = -1$), which should determine $u$ there.

Note: A system like (8) arises in the study of the stability of self-sustaining traffic jams ("jimitons").

Example: The wave equation $u_{tt} - c^2 u_{xx} = 0$ can be written as a system in terms of the variables $\varphi = u_t - cu_x$, $\psi = u_t + cu_x$.

Then $\varphi_t + c\varphi_x = 0$ \hspace{1cm} (9)

$\psi_t - c\psi_x = 0$ \hspace{1cm} (10)

Thus $\varphi = \varphi(x - ct)$ and $\psi = \psi(x + ct)$. Now suppose that we know $u$ and $u_t$ at $t = 0$:

$a(x) = u(x, 0)$ \hspace{1cm} (11)

$b(x) = u_t(x, 0)$
Hence, evaluating at \( t=0 \)

\[
\begin{align*}
\varphi(x) &= b(x) - c a'(x) \\
\psi(x) &= b(x) + c a'(x)
\end{align*}
\]

\[
\Rightarrow u_t = \frac{1}{2} \left\{ \varphi(x-ct) + \psi(x+ct) \right\}
\]

\[
= \frac{1}{2} \left\{ b(x-ct) + b(x+ct) - c a'(x-ct) + c a'(x+ct) \right\}
\]

Integrating

\[
u = \frac{1}{2} \left\{ a(x-ct) + a(x+ct) \right\} + \frac{1}{2c} \int_{x-ct}^{x+ct} b(s) \, ds + \alpha(x)
\]

Evaluate at \( t=0 \) \( \Rightarrow \alpha \equiv 0 \)

This is D'Alembert's solution to the wave equation \( \{ -\infty < x < \infty \}, \text{no boundaries} \)

Note that, unlike the simple 1st order scalar case, the domain of dependence for \( (x,t) \) in the full interval \( [x-ct, x+ct] \) at \( t=0 \). Not just the end points.