Addendum to Lecture 12

Principal value: Let \( \varphi = \varphi(x) \) be a test function, \( -\infty < x < \infty \). The principal value arises from the desire to give meaning to the integral
\[
\int_{-\infty}^{\infty} \frac{\varphi(x)}{x} \, dx \quad \text{I} \quad (A.1)
\]

This integral is not properly defined because it behaves like \( \varphi(0)/x \) near \( x=0 \), and \( 1/x \) is not integrable if \( \varphi(0) \neq 0 \), then there is no problem, because then \( \varphi(x) \sim \varphi'(0)x \) for \( x \) small.

*Way #1: Write* \( \varphi = \varphi_e + \varphi_o \) \((A.2)\)

where \( \varphi_e = \frac{1}{2}(\varphi(x) + \varphi(-x)) \) is the even part

*As to why one would like to do this... I will give you some reasons later.* See p12.21 and beyond
of \( \Phi \) and \( \Phi_0 = \frac{1}{2}(\Phi(x) - \Phi(-x)) \) is the odd part. Then argue that it should be \( \int_{-\infty}^{\infty} \frac{1}{x} \Phi_0(x) \, dx = 0 \) (A.3)

because the integrand is odd and thus write

\[
I = \int_{-\infty}^{\infty} \Phi(x) \, dx = \int_{-\infty}^{\infty} \Phi_0(x) \, dx
\]

which is well defined because \( \Phi_0 = O(x) \) for \( x \) small.

This is the **Principal Value** and it is indicated by the symbol \( \text{P.V.} \) (A.5)

Way #2

Define

\[
\int_{-\infty}^{\infty} \frac{\Phi(x)}{x} \, dx = \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{\Phi(x)}{x} \, dx
\]

(A.6)
Let us show this is the same as (A.4)

\[
\int_{-\infty}^{\infty} \varphi(x) \frac{dx}{x} = \int_{-\infty}^{-\epsilon} \varphi(x) \frac{dx}{x} + \int_{-\epsilon}^{\epsilon} \varphi(x) \frac{dx}{x} + \int_{\epsilon}^{\infty} \varphi(x) \frac{dx}{x} = \int_{-\infty}^{\infty} \varphi_0(x) \frac{dx}{x}
\]

Clearly \( I_2 + J_2 = 0 \), while \( I_1 + J_1 \)

converge to the formula in (A.4) because

\( \varphi_0(x)/x \) is integrable. Q.E.D.

Way #3: Let \( \phi(x) \) be an even test function

with \( \phi(0) = 1 \) and write

\[
I = \int_{-\infty}^{\infty} \left( \frac{1}{x} \right) (\phi(x) - \phi(0)\phi(x)) \, dx + \phi(0) \int_{-\infty}^{\infty} \phi(x) \, \frac{dx}{x}
\]

\( \frac{1}{x} \) integrable

(A.7) Argue this is zero

because the numerator is odd
Again, this is the same as (A.4) and the answer does not depend on $\phi$.

Proof \[
\frac{1}{x} [\varphi(x) - \varphi(0) \Phi] = \frac{1}{x} \varphi_0 + \frac{1}{x} \left[ \varphi_e(x) - \varphi(0) \Phi \right] \]

\[
\leq K
\]

However \( \varphi_e(0) = \varphi(0) \) \( \therefore K \) is integrable and odd \( \therefore \int K dx = 0 \).  \text{Q.E.D.}

Attempts using "naive" approach fail; shown next.

Generalizations of the principal value

The approach used to define $f$ earlier works because the singularity $1/x$ has an "area" under it, but

this area in the difference

between two "equal" infinities, and we can cancel the infinity by $1/x$

Let us call it the "naive" approach.
"proper" limit, as in (A.6)

Note, however, that the value depends on which
limit. For example \( \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} q(x) \frac{dx}{2\varepsilon} \) gives a different answer.

But the symmetric limit has nice properties!

But this fails for higher order singularities

For example \( \frac{1}{x^2} \) is non-integrable and the infinity is all of the same sign. Thus suppose we

(i) Try approach leading to (A.9)? We would then argue that \( \int_{\frac{1}{x^2}} q_0 \, dx = 0 \) because the integrand is odd and this yields

\[
\int_{\frac{1}{x^2}} q \, dx = \int_{\frac{1}{x^2}} q_0 \, dx
\]  \( (A.9) \)

but this is useless
because $\frac{1}{x^2} \varphi(x)$ is not integrable, except for the special case when $\varphi(x) = \varphi(\phi)$.

(ii) Try the approach leading to (A.6)?

Does not work because any limit of the form
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{s} \varphi(x) \, dx, \quad s = s(\varepsilon)
\]
will diverge no matter how $s \downarrow 0$ as $\varepsilon \downarrow 0$

There is no finite cancellation!

Analogy: The series $\sum_{n=1}^{\infty} (-1)^n/n$ can be reordered to converge to any value in $[-\infty, \infty]$. But $\sum_{n=1}^{\infty} 1/n$ diverges no matter the order.

(iii) Suppose we try the approach in (A.7)?

Then we would write.
\[
\int_{-\infty}^{\infty} \frac{1}{x^2} \varphi(x) \, dx = \int [\varphi(x) - \varphi(0) \phi_1(x) - \varphi'(0) \phi_2(x)] \frac{dx}{x^2} + \frac{I_1}{I_2} \int \frac{\varphi(x) \phi_1(x) \, dx}{x^2} + \frac{I_1}{I_3} \int \frac{\varphi'(0) \phi_2(x) \, dx}{x^2}
\]

where \( \phi_1 \) is an even test function with \( \phi_1(0) = 1 \)
\( \phi_2 \) is odd \( \phi_2'(0) = 1 \)

Then \( I_1 \) is well defined.

\( I_3 \) can be argued to vanish because the integrand is odd

\( I_2 \leftarrow ?? \)

Of course, you could say: well let us throw out \( I_2 \) and \( I_3 \) (the infinities) and keep only the finite part \( I_1 \) !

_problems is:_ The answer depends on
the choice of $\phi_1$! (A.12)

and there is no clear "special" $\phi_1$

Example $\frac{1}{x^3}$. OK, here there is also a situation with two infinities "equal and of opposite sign. So maybe the naive approach work?"

Answer: No. $\frac{1}{x^3}$ $\phi(x) \sim \frac{1}{x^3}\phi(0) + \frac{1}{x^2}\phi'(0)$

$+ \frac{1}{2x}\phi''(0) + \text{regular}$

and the $\frac{1}{x^2}$ singularity ruins it. (A.12) b

Generalizations of the principal value using generalized derivative notions

Earlier (pp 12.03 - 12.04) we saw that the generalized derivative of $\ln|1-x|$ was the principal value; that is:
\[ \int_{-\infty}^{\infty} (\ln|x|) \varphi(x) \, dx = \int_{-\infty}^{\infty} \frac{1}{x} \varphi(x) \, dx \quad (A.13) \]

We can now use this idea to define:

\[ \int_{-\infty}^{\infty} \frac{1}{x^2} \varphi(x) \, dx \overset{\text{Definition}}{=} \int_{-\infty}^{\infty} (\ln|x|) \varphi'(x) \, dx \quad (A.13) \]

\[ \begin{align*}
\text{Generalized derivative manipulations} & \quad \left\{ \begin{array}{c}
\int = -\int (\ln|x|)'' \varphi(x) \, dx \\
\int = -\int (\ln|x|) \varphi''(x) \, dx
\end{array} \right. \quad (A.14)
\end{align*} \]

Unfortunately (A.14) does not have a nice "intuitive" interpretation as "keep the finite part and delete the \( \infty \) which the naive approach provides." \( ^{+} \)  

\[ ^{+} \text{At least not that I know.} \]
On the other hand this approach allows defining many other singular integrals. For example:

$$\int_{-\infty}^{\infty} |x|^{-\alpha} \varphi(x) \, dx = \int_{-\infty}^{\infty} \frac{\sigma |x|^{1-\alpha}}{\alpha-1} \varphi'(x) \, dx$$

$$\sigma = \text{sign} \, x$$

$$= \int_{-\infty}^{\infty} \sigma |x|^{1-\alpha} \varphi'(x) \, dx$$

since

$$\frac{d}{dx} \sigma |x|^{1-\alpha} = -|x|^{-\alpha}$$

**None-test functions**

Note that the P.V. can be defined even if \( \varphi \) is not smooth! (A.6) requires

(i) \( \frac{1}{x}[\varphi(x)-\varphi(0)] \) integrable near zero

(ii) \( \frac{1}{x} \varphi(x) \) integrable at \( 0 \)

It can also be defined over finite
and \( f \) vanishes as \( |z| \to \infty, \text{Re}(z) > 0 \).

Intuitively - you just need to keep track of the boundary contributions when integrating by parts!

The Hilbert Transform

Principal Values and Analytic Functions

Let \( \varphi = \varphi(x) \) be a test function \(^\dagger\) for \(-\infty < x < \infty\). Is there a \( \psi = \psi(x) \) such that \( \varphi + i\psi \) is the limit on the real axis of a function \( f = f(z) \) analytic for \( y > 0 \)? (Here \( z = x + iy \)).

The answer is \textbf{yes}. We show this next.

Define

\[
\hat{f}(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x-z} \, dx \qquad \text{Re}z > 0
\]

\(^\dagger\) This can be done for \( \varphi \)'s that need not be as smooth as test functions, nor decay like them.
Then \( f \) is analytic for \( \text{Re}(z) > 0 \) and vanishes as \( |z| \to \infty \). We now compute

\[
\lim_{z \to 0} f(z) = \begin{cases}
\text{the limit } z \to 0 \text{ for any } x_0
\end{cases}
\]

\[
f(z) = \frac{1}{i\pi} \int_{-\infty}^{e} \frac{\phi(x) \, dx}{x-z} + \frac{1}{i\pi} \int_{e}^{\infty} \frac{\phi(x) \, dx}{x-z}
\]

\[
+ \frac{1}{i\pi} \int_{-e}^{e} \frac{\phi(0) \, dx}{x-z} + \frac{1}{i\pi} \int_{-e}^{e} \frac{\phi(x) - \phi(0) \, dx}{x-z}
\]

where \( 0 < e < 1 \)

Now \( J \) can be computed explicitly, using the principal branch of \( \log \):

\[
J = \frac{1}{i\pi} \left\{ \log(e-z) - \log(-e-z) \right\}
\]

\[
= \frac{1}{i\pi} \left\{ \ln r_2 - \ln r_1 + i(\theta_2 - \theta_1) \right\}
\]
Hence the limit is

$$\lim_{z \to 0} f(z) = \frac{1}{i\pi} \int_{-\infty}^{-\epsilon} \frac{\varphi(x) \, dx}{x} + \frac{1}{i\pi} \int_{\epsilon}^{\infty} \frac{\varphi(x) \, dx}{x}$$

$$+ \varphi(0) + \frac{1}{i\pi} \int_{-\epsilon}^{\epsilon} \frac{\varphi(x) - \varphi(0) \, dx}{x}$$

Hence, taking now $\epsilon \to 0$ we see that

$$\lim_{z \to x_0} f(z) = \varphi(x_0) + \frac{1}{i\pi} \int \frac{\varphi(x) \, dx}{x-x_0}$$

(A.20)

Thus

$$\psi(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(s) \, ds}{s-x}$$

$$= H \varphi$$  (A.21)

$H$ is called the Hilbert transform.

There is an analogous problem for periodic functions (say, of period $2\pi$). It can be stated as follows:
Given a real valued function on the unit circle \( |z| = 1 \), \( \varphi = \varphi(\theta) \), \( z = e^{i\theta} \), find another function \( \psi = \psi(\theta) \) such that \( \varphi + i\psi \) is the limit on \( |z| = 1 \) of an analytic function \( f(z) \) defined for \( |z| < 1 \).

Again \( \psi = H \varphi \) (A.23)

where \( H \) is the periodic Hilbert Transform also given by a principal value integral.

To make \( \psi \) unique require \( \int_0^{2\pi} \varphi(\theta) d\theta = 0 \) (A.23)

**Hilbert Transform and Fourier Series**

Let \( \varphi(\theta) = \sum_{n=-\infty}^{\infty} \varphi_n e^{in\theta} \)

\( \varphi_n = \varphi_{-n} \) (A.25)
Note: \( \phi \) smooth \( \Leftrightarrow \phi_n \) decays fast with \(|x| \to \infty\)

Then
\[
\phi(\theta) = \phi_0 + 2 \text{Re} \sum_{n=1}^{\infty} \phi_n e^{in\theta} \\
\hat{f}(z) = \phi_0 + 2 \sum_{n=1}^{\infty} \frac{\phi_n}{z^n}; z = r e^{i\theta} \\
\psi(\theta) = 2 \text{Im} \sum_{n=1}^{\infty} \phi_n e^{in\theta} \tag{A.26}
\]

Note \( \psi = \sum \phi_n e^{in\theta} \); \( \phi_n = -i \text{sign}(n) \phi_n \)

\{ Hilbert transform in Fourier \}
\( \phi_0 = 0 \), of course.

\begin{align*}
\text{Hilbert Transform and Fourier Transform} \\
\phi &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(k) e^{ikx} \, dk = \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} \hat{\phi}(k) e^{ikx} \, dk \right) \\
\hat{f}(z) &= \frac{1}{\pi} \int_{0}^{\infty} \hat{\phi}(k) e^{ikz} \, dk \\
\psi &= \frac{1}{\pi} \text{Im} \int_{0}^{\infty} \hat{\phi}(k) e^{ikx} \, dk \\
\hat{\psi}(k) &= -i \text{sign}(k) \hat{\phi}(k) \tag{A.28}
\end{align*}

This is \( (A.21) \)
The Dirichlet to Neumann Map is as follows:

Solve an equation, say \( \Delta u = 0 \), in some domain \( \Omega \), with Dirichlet boundary conditions: \( u = f \) on \( \partial \Omega \)

where \( f \) is some function defined on \( \partial \Omega \). Then let \( g = \frac{\partial u}{\partial n} \) be the normal derivative of the solution along \( \partial \Omega \). Then \( g \) is a function defined on \( \partial \Omega \) and \[ g = DTN(f) \]

That is, \( DTN \) is an operator mapping functions defined on \( \partial \Omega \) to functions in \( \partial \Omega \).
Consider the example of the unit disk.

\[ f = \sum_{n=0}^{\infty} f_n e^{i n \theta} \]

Note: \( f \) and \( u \) real

\[ u = f(\theta) \]

\[ \Delta u = 0 \]

Thus

\[ u = f_0 + 2 \text{Re} \sum_{n=1}^{\infty} f_n e^{i n \theta} \]

Using that \( f_{-n} = \overline{f_n} \)

\[ g = \text{DTN}(f) = u_{r=1} \]

\[ = 2 \text{Re} \sum_{n=1}^{\infty} n f_n e^{i n \theta} \]

\[ = 2 \text{Im} \sum_{n=1}^{\infty} \bar{n} f_n e^{i n \theta} \]

On the other hand

\[ f' = 2 \text{Re} \sum_{n=1}^{\infty} \bar{n} f_n e^{i n \theta} \]

\[ \text{Note} \quad u = \text{Re} \left\{ f_0 + 2 \sum f_n z^n \right\}, \ z = re^{i\theta} \]

and real part of an analytic function harmonic

\[ \Rightarrow \]

\[ g = \text{DTN}(f) = u_{r=1} \]

\[ = 2 \text{Re} \sum_{n=1}^{\infty} n f_n e^{i n \theta} \]

\[ = 2 \text{Im} \sum_{n=1}^{\infty} \bar{n} f_n e^{i n \theta} \]

\[ \text{On the other hand} \]

\[ f' = 2 \text{Re} \sum_{n=1}^{\infty} \bar{n} f_n e^{i n \theta} \]

\[ \text{Note} \quad u = \text{Re} \left\{ f_0 + 2 \sum f_n z^n \right\}, \ z = re^{i\theta} \]

and real part of an analytic function harmonic.
Now compare with (A.26) where the Hilbert Transform \( \Phi \rightarrow \psi \) for periodic functions is given.

Clearly

\[
q = DTN(f) = Hf'
\]

which relates the Hilbert Transform to the DTN.

Water waves and the DTN

The equations for "water waves" with the following approximations

\( z = b(x) \)

\( \vec{u} = \nabla \phi \)

(1) Constant density \( \rho \)

(2) Ignore air \( \rho_0 \ll \rho \)
3. Inviscid and irrotational flow: \( \vec{u} = \nabla \phi ; \phi = \phi(x, z) \)

4. Ignore surface tension

\( \begin{align*}
\text{(i)} \quad \Delta \phi &= 0 \quad \text{for} \quad b < z < \gamma \\
\text{(ii)} \quad \hat{n} \cdot \nabla \phi &= 0 \quad \text{for} \quad z = b \\
&\quad \text{(impermeable bottom)} \\
\text{(iii)} \quad \gamma_t + \phi_x \phi_z &= \phi_z \quad \text{for} \quad z = \gamma \quad \text{Kinematic BC} \\
\text{(iv)} \quad g \gamma + \phi_t + \frac{1}{2} \phi_x^2 &= 0 \quad \text{for} \quad z = \gamma \quad \text{Dynamic BC} \\
&\quad g = \text{acceleration of gravity}
\end{align*} \)

In 3-D replace \( \phi_x \rightarrow \nabla_h \phi, \phi_z \rightarrow \nabla_h \gamma \)

where \( \nabla_h = (\partial_x, \partial_\gamma) \)

Now define \( \varphi = \varphi(x, t) = \phi(x, \gamma, t) \)

\( \begin{align*}
\hat{n} &= \text{unit normal to surface} \\
\psi &= (\hat{n} \cdot \nabla \phi)\bigg|_{z = \gamma} = DTN(\varphi)
\end{align*} \)
We are going to show now that, using the DTN, the equations can be written as equations for \( y = y(x,t) \) and \( \phi = \phi(x,t) \) only.

Note \( \hat{n} = (-\nu_x, 1) \):

\[
\psi' = -\nabla_x \phi^t + \phi_z \bigg|_{z = y}
\]

Chain rule \( \Rightarrow \)

\[
\phi_t = \phi_z \nu_t + \phi_t \bigg|_{z = y}
\]

\[
\phi_x = \phi_x + \phi_z \nu_x \bigg|_{z = y}
\]

\[
\phi_z \quad \text{(A.35)}
\]

This is a linear system, with \( \det = (1 + \nu_x^2) \), which we can solve to write \( \phi_t, \phi_x, \phi_z \) at the surface in terms of \( y, \phi, \psi \).
Substituting into (iii-iv) of (A.33) then yields the desired equations.

**Example:** Linear case (infinitesimal slab.)

Then, on \( z = y \): \( \Phi_z = \Psi, \Phi_x = \Psi_x, \Phi_t = \Psi_t \)

and the equations are

\[
\eta_t = \Psi; \quad \eta_y + \Psi_t = 0; \quad \Psi = \text{DTN} \Psi
\]

(A.36)

Advantage: the DTN can be computed using Boundary Integral Methods (BIM) and there is no need to evaluate/compute the values of \( \Psi \) everywhere in the domain.

**Note** that the DTN also relates the temperature on the boundary with the heat flux (for steady state).

End Appendices follow.
Appendix A

Example: principal value integral for a function with a continuous derivative, on a finite interval. (Below $a < 0 < b$)

\[
\int_{a}^{b} \frac{1}{x} f(x) \, dx = \ln|x| \bigg|_{a}^{b} - \int_{a}^{b} (\ln|x|) f'(x) \, dx
\]

(Ap1) \[ (\ln b) f(b) - (\ln a) f(a) \]

If \( f \) has a second derivative, then

\[
\int_{a}^{b} \frac{1}{x^2} f(x) \, dx = \left( -\frac{1}{x} f(x) \right) \bigg|_{a}^{b} + \int_{a}^{b} \frac{1}{x} f'(x) \, dx
\]

(Ap2) \[ -\frac{1}{b} f(b) + \frac{1}{a} f(a) \]

† You can arrive at the same answer by writing \( f = f_o + f_e \) (odd and even parts) and arguing that \( \int_{-S}^{S} \frac{1}{x} f_e \, dx = 0 \) for any \( S > 0 \). But this argument does not work for (Ap2).