Lecture 12, Tu March 31 2020

Shocks and weak solutions

Can we give meaning to \( p_t + g_x = 0 \) when there are shocks?

That is: what do \( p_t \) and \( g_x \) mean when \( p \) and \( g \) are discontinuous?

Answer: yes

Single variable

\[ \phi \text{ smooth and vanishing as } |x| \to \infty \]

\( \phi^{(n)} \text{ vanishes "very fast" for all } n \)

If \( f = f(x) \) has a derivative, then

\[
\int_{-\infty}^{\infty} f'(x) \phi(x) \, dx = -\int_{-\infty}^{\infty} f(x) \phi'(x) \, dx
\]

(1)

for any \( \phi \) \{ "Test Functions" \}

Vice versa

\[
\int f' \phi \, dx = \int g \phi \, dx
\]

for all \( \phi \iff g = f \)

(2)
Use (1) to define derivative!

(3) Function $f \iff$ operator $D_f = \int f \phi \, dx$

(4) Derivative operator

\[ D_f^1 [\phi] = -D_f [\phi'] \]

This is (1)

\[ \therefore D_f^1 = D_f \iff f' \text{ exists} \]

(5) Can use (4) to define derivative for any $f$ that defines a $D_f$

Do not need $f$ differentiable.

Example

\[ H = \begin{cases} 
0 & \text{for } x < 0 \\
1 & \text{for } x > 1 
\end{cases} \] (Heaviside function)

\[ \int_{-\infty}^{\infty} H'(x) \phi(x) \, dx = -\int_{-\infty}^{\infty} H(x) \phi'(x) \, dx = \phi(0) \]

\[ \Rightarrow H = \delta \]
\[ \int_{-\infty}^{\infty} H'' \varphi \, dx = -\int H' \varphi' \, dx = \int H \varphi'' \, dx \]

This is \( S' \) \( = -\varphi'(a) \)

Note agrees with "smear-up limit"

\[
\begin{array}{c}
\text{H} \\
\text{\rightarrow} \quad \text{H} \\
\text{\leftarrow e} \\
\text{area} 1 \\
\text{H}'' \\
\end{array}
\]

\[ \lim_{\epsilon \to 0} \int \text{(smearred} H') \varphi \, dx = \varphi(a) \]

etc.

**Example**

Let \( f = \ln |x| \)

Then \( f' = \frac{1}{x} \) not integrable

So what is

\[ \mathcal{D} \left[ f \right] = \int \frac{1}{x} \varphi \, dx = -\int \ln |x| \varphi' \, dx \]
\[- \int_{-\infty}^{\infty} (\ln |x|) \varphi'(x) \, dx = -\lim_{e \to 0} \int_{-e}^{e} (\ln |x|) \varphi'(x) \, dx \]

\[\lim_{e \to 0} \int_{-e}^{e} \frac{1}{x} \varphi \, dx + \ln e \varphi(e)\]

\[= -\ln e \varphi(e) + \int_{-\infty}^{\infty} \frac{1}{x} \varphi \, dx + \ln e \varphi(e)\]

\[+ \int_{e}^{\infty} \frac{1}{x} \varphi \, dx = \int_{e}^{\infty} \frac{1}{x} \varphi \, dx\]

That is: \[(\ln |x|)^1 = \text{principal value} \frac{1}{x}\]

Example Let \(f\) be continuous near \(a\) and \(b\) then define

\[\int_{a}^{b} f'(x) \, dx = \lim_{\varphi \to \text{square function}} \int_{a}^{b} f' \varphi \, dx\]

limit over \(\varphi\)'s

approach square function
Then \[ \int_{a}^{b} f'(x) \, dx = -\lim_{\varphi \to 0} \int_{a}^{b} \varphi' \, dx = \varphi(b) - \varphi(a) \]

because \[ \text{area}_1 \]

\[ \text{area}_2 \]

**Fundamental theorem of calculus**

applies.

Unfortunately, the **product rule** does not apply in general.

If I only have \( D_f \) and \( D_g \), what is \( D_{fg} \)?

Not even defined.
Example: what is \( S^2 \)??

\[
\int S^2 \varphi(x) \, dx = S(x) \varphi(x) \bigg|_{x=0} = ?
\]

However, if \( f \) is smooth, then \( h = f g \) well defined:

\[
D_h[\varphi] = D_g[f \varphi] = \int f g \varphi \, dx
\]

Even if \( g \) is not a function

Then product rule works

\[
D_{h'} = D_{f'g} + D_{fg'}
\]

\[
D_{h'}[\varphi] = -D_{h'}[\varphi'] = -D_g[f \varphi']
\]

\[
D_{f'g}[\varphi] = D_g[f' \varphi]
\]

\[
D_{fg'}[\varphi] = D_g[f \varphi'] = -D_g[&(f \varphi')']
\]
Also chain rule not clear
ingeneral. Example what is \( S(S(x)) \)?

But can define \( f(g(x)) \)
when \( g \) smooth, \( g' > 0 \),
and \( g \to \pm \infty \) as \( x \to \pm \infty \)

\[ \text{Skep details} \]

If \( \chi(x) \) inverse \( g \) \[
[ g(\chi(x)) = x ]
\]

Then \( h = f(g) \) given by

\[ D_h \[ \varphi \] = D_f \[ \varphi(\chi(x)) \chi'(x) \] \]

Note \[
\int f(g(x)) \varphi(x) \, dx = \int f(g) \varphi(x(g)) \chi'(g) \, dg
\]

Leave to reader show chain rule applies
Fourier Series/Transforms

Let us do \( f \) (periodic test functions, then)

For \( D_f \) write \( f = \sum f_n e^{i n \theta} \)

where \( f_n = \frac{1}{2\pi} D_f[e^{-i n \theta}] \)

\( f \) need not be a function. Now

\[ \int_0^{2\pi} \sum f_n e^{i n \theta} \varphi(\theta) \, d\theta = \sum f_n \varphi_n \]

where \( \varphi = \sum \varphi_n e^{i n \theta} \) (\( \varphi \) smooth, so converges great)

On the other hand

\[ D_f[\varphi] = D_f[\sum \varphi_n e^{i n \theta}] = \sum \varphi_n D_f[e^{i n \theta}] = \sum \varphi_n f_{-n} \] so \[ \text{WORKS} \]
Example: periodic S function

Then

$$S(x) = \sum \frac{1}{2\pi} e^{in\theta}$$

Also note

Fournier Worf $S^1$, \( \frac{1}{2\pi} D_x [e^{-in\theta}] = \frac{in}{2\pi} \)

Term by term differentiation works.

So now you know what \( \sum_{n \neq 0} np e^{in\theta} \)

means ...

Example: What is

$$\left\{ \sum_{n \neq 0} \frac{1}{2\pi} e^{in\theta} \right\} = S(x)$$

Well \( S' = S - \frac{1}{2\pi} \) and \( S \) has mean zero.
\[ S(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ -1 & \text{for } -\pi < x < 0 \end{cases} \]

Sawtooth

Integrate once again

(and keep mean = 0)

\[
\sum_{\eta \neq 0} \frac{-1}{2\pi\eta^2} e^{i\eta\theta} = \sum_{\eta \neq 0} \frac{-1}{2\pi\eta^2} e^{i\eta\theta}
\]

Evaluate at \(\theta = 0\) \(\Rightarrow\) \[\sum_{\eta = 1}^{\infty} \frac{1}{\eta^2} = \frac{\pi^2}{6}\]

at \(\theta = \pi\) \(\Rightarrow\) \[\sum_{\eta = 1}^{\infty} \frac{(-1)^{\eta+1}}{\eta^2} = \frac{\pi^2}{12}\]

Integrate again to get \(1/\eta^3\) c.c.m.s. etc

Finally \(u = H(x-\sigma(t))\) smooth

\[
\Rightarrow u_x = \delta(x-\sigma) \quad u_t = -\sigma \delta(x-\sigma) \quad \text{chain rule}
\]