# Lecture 12
## The Laplace Method

We begin with integrals of the form

$$I(\lambda) = \int_a^b e^{-\lambda v(x)} h(x) dx, \tag{8.8}$$

where $v(x)$ and $h(x)$ are independent of the parameter $\lambda$. The variable of integration $x$ is real. We shall show how to find the asymptotic form of $I(\lambda)$ as $\lambda \to \infty$.

We note that the integrand in the integral above is maximum when $v$ is minimum. Indeed, let $x_0$ be the point inside the interval $[a, b]$ at which $v(x)$ is minimum. Then the integrand at $x_0$ is exponentially larger than that at any other points in the region of integration. As a result, to evaluate the integral of (8.8) approximately, it is crucial to approximate $v(x)$ and $h(x)$ accurately near $x$. This also means that we need not be concerned with the integrand away from $x_0$. This is because the dominant contribution to the integral comes from a very small region near $x_0$.

### Problem for the Reader:

Find the leading asymptotic term of the integral

$$I(\lambda) = \int_0^\infty \exp(-\lambda \sinh^2 x) dx, \lambda \gg 1. \tag{8.9}$$

### Answer

Comparing (8.9) with (8.8), we make the identification

$$v(x) = \sinh^2 x,$$

and

$$h(x) = 1.$$

The function $\sinh^2 x$ is always positive, and is minimum at $x = 0$. Thus it suffices to represent $\sinh^2 x$ accurately near the origin, where

$$\sinh^2 x \approx x^2.$$

We therefore have

$$I(\lambda) \approx \int_{-\infty}^\infty e^{-\lambda x^2} dx.$$

Let

$$\rho \equiv \sqrt{\lambda}\, x,$$

and

$$dx = d\rho/\sqrt{\lambda}\,.$$

With this change of variable the exponent is independent of $\lambda$. Then we have

$$I(\lambda) \approx \frac{1}{\sqrt{\lambda}} \int_{-\infty}^\infty e^{-\rho^2} d\rho = \sqrt{\frac{\pi}{\lambda}}\,. \tag{8.10}$$

In the above, we have made use of the fact that the Gaussian integral $\int_{-\infty}^\infty e^{-\rho^2} d\rho$ is equal to $\sqrt{\pi}$, as is given in Appendix A of this chapter.

If one makes a small mistake in an approximation, the answer one gets can be far off the mark. Thus it is often useful to have a quick answer. We shall show how to get a rough estimate of the asymptotic form of the integral $I(\lambda)$ without going through a great deal of calculations performed.

The integrand of $I(\lambda)$ drops by a factor of $e$ when

$$\lambda x^2 = 1,$$

or

$$x = 1/\sqrt{\lambda}\,.$$

Thus $1/\sqrt{\lambda}$ is roughly the width of the region which gives the dominant contributions to the integral. The width $1/\sqrt{\lambda}$ of the region of dominant contribution is the same factor relating $dx$ with $dt$. We shall say that the scale of $x$ is $1/\sqrt{\lambda}$.

Asymptotic Expansions of Integrals

The integral $I(\lambda)$ is roughly equal to its integrand at $x = 0$ times the width of the region of dominant contributions. Since the integrand at $x = 0$ is equal to unity, $I(\lambda)$ is of the order of $1/\sqrt{\lambda}$ . This produces the answer (8.10) up to a multiplicative constant. We express this estimate as

$$I(\lambda) = O\left(1/\sqrt{\lambda}\right).$$

For certain problems such an estimate is already adequate for the purpose. If so, we are spared the chore of a tedious calculation. The factor missing in the estimate is the multiplicative constant $\sqrt{\pi}$, which is the value of the Gaussian integral.

Homework problems in Chapter 8 due Nov 1,04:

Problem 1. Find the leading term (not just an estimate) for each of the three integrals on p. 218.

Problem 2. Find the leading term for each of the following integrals

**a**.

$$\int_{-1}^{1} e^{-\lambda t^3} dt,$$

**b**.

$$\int_{1}^{\infty} e^{-\lambda t^2} dt,$$

**c**.

$$\int_{-2}^{1} \sin x e^{-\lambda x^2} dx,$$

**d**.

$$\int_{-\pi}^{\pi} e^{-\lambda \sin x} dx,$$

**e**.

$$\int_{0}^{\infty} e^{-\lambda x} e^{-x^2} dx,$$

**f**.

$$\int_{0}^{\lambda} e^{t^3} dt,$$

**g**.

$$\int_{0}^{\infty} e^{-\lambda(t+t^5)} dt.$$

Problem 3. Find the entire asymptotic series for each of the integrals in problem 2.