

# Lectures Nine and Ten The WKB Approximation

The WKB method is a powerful tool to obtain solutions for many physical problems. It is generally applicable to problems of wave propagation in which the frequency of the wave is very high or, equivalently, the wavelength of the wave is very short. The WKB solutions are approximate solutions, but sometimes they are surprisingly accurate. In this chapter we'll discuss this method, which is applicable to linear equations only.

## A The Zeroth and the First-Order WKB

We consider the second-order differential equation

$$y'' + p^2 y = 0. \quad (7.1)$$

If  $p$  is a constant, the two independent solutions of (7.1) are  $\exp(\pm ipx)$ , waves with wave number  $p$  travelling in opposite directions of the  $x$ -axis.

If  $p$  is a function of  $x$ , it appears reasonable that the solutions are two waves with the phase  $\pm \int p(x) dx$ . Thus we may surmise that the independent solutions of (7.1) are

$$e^{\pm i \int p(x) dx}, \quad (7.2)$$

which are the zeroth-order WKB solutions.

Let us see if these solutions satisfy (7.1). We have

$$\left( \frac{d^2}{dx^2} + p^2 \right) e^{\pm i \int p(x) dx} = \pm ip' e^{\pm i \int p(x) dx} \neq 0. \quad (7.3)$$

Therefore, the WKB solutions (7.2) do not satisfy (7.1) unless  $p' = 0$ , or  $p$  is independent of  $x$ .

While we get a negative answer, (7.3) suggests that  $\exp(\pm i \int p(x) dx)$  are good approximate solutions of (7.1) provided that  $\pm ip'$  is negligible, or, more precisely, if

$$|p'| \ll p^2,$$

the left-side of this inequality being a term inside the parenthesis of (7.3). The inequality above can be written as

$$\left| \frac{d}{dx} \frac{1}{p} \right| \ll 1. \quad (7.4)$$

This condition is satisfied if  $p(x)$  is of the form

$$p(x) \equiv \lambda P(x), \quad (7.5)$$

where  $\lambda$  is a large constant, i.e.,

$$\lambda \gg 1, \quad (7.6)$$

and  $P(x)$  is of the order of unity. Indeed, if  $p(x)$  is given by (7.5), the inequality (7.4) is

$$\left| \frac{1}{\lambda} \frac{d}{dx} \frac{1}{P(x)} \right| \ll 1. \quad (7.7)$$

Clearly, (7.7) is satisfied if  $\lambda \gg 1$ , provided that  $x$  is not near a zero of  $P(x)$ .

We note that the phases of the solutions  $\exp(\pm i \int p(x) dx)$  are functions of  $x$ , but the magnitudes of these approximate solutions are independent of  $x$ . Let us remember that, in Chapter 1, we have shown that if  $y_1$  and  $y_2$  are two independent solutions of (7.1), then the Wronskian  $W(x) \equiv y_1 y_2' - y_1' y_2$  is independent of  $x$ . Now the Wronskian of  $\exp(i \int p(x) dx)$  and  $\exp(-i \int p(x) dx)$  is easily shown to be equal to  $2ip(x)$ , not a constant unless  $p(x)$  is a constant. This suggests that these approximate solutions still leave something to be desired. Because the Wronskian of the approximate solutions miss by a factor  $p(x)$ , let us try to fix it by adding an additional factor  $1/\sqrt{p(x)}$  to each of the approximate solutions. The resulting approximate solutions are

$$y_{WKB}^{\pm}(x) = \frac{1}{\sqrt{p(x)}} e^{\pm i \int p(x) dx}. \quad (7.8)$$

which includes both the zeroth order and the first order terms of the WKB approximations.

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The magnitude of these solutions varies with  $x$  like  $1/\sqrt{p(x)}$ . The Wronskian of  $y_{\text{WKB}}^{\pm}(x)$  is now exactly a constant. (See homework problem 1.) It is therefore tempting to surmise that, under the condition (7.4) or equivalently, (7.7),  $y_{\text{WKB}}^{\pm}(x)$  are even better approximations than  $e^{\pm i \int p(x) dx}$ .

To see if this is true, we put

$$y \equiv e^{\pm i \int p(x) dx} v, \quad (7.9)$$

and substitute this expression of  $y$  into (7.1). We get

$$(D \pm ip)(D \pm ip)v + p^2 v = 0.$$

or

$$\left(\frac{d^2}{dx^2} \pm 2ip \frac{d}{dx} \pm ip'\right)v = 0.$$

We shall write the equation above as

$$v' + \frac{P'}{2P}v = \pm \frac{i}{2P}v''. \quad (7.10)$$

By (7.5), (7.10) can be written as

$$v' + \frac{P'}{2P}v = \pm \epsilon \frac{i}{2P}v'', \quad (7.11)$$

where

$$\epsilon \equiv 1/\lambda$$

is a small number. In the first-order approximation, we ignore the right side of (7.11) and we get

$$v' + \frac{P'}{2P}v \approx 0, \quad (7.12)$$

which gives

$$v(x) \approx \frac{1}{\sqrt{P(x)}}. \quad (7.13)$$

Thus (7.9) and (7.13) give, aside from an immaterial overall constant, the WKB solutions (7.8).

We have mentioned that if  $P(x)$  has a zero at  $x_0$ , the inequality (7.10) does not hold at  $x_0$ . To see how far away from  $x_0$  it must be for the WKB approximation to hold, let  $P(x)$  near  $x_0$  be approximately given by

$$P(x) \approx a(x - x_0)^n, \quad x \approx x_0. \quad (7.14)$$

Then (7.7) requires

$$|x - x_0| \gg \left(\frac{n}{\lambda a}\right)^{1/(n+1)}. \quad (7.15)$$

Eq. (7.15) tells us how far away from  $x_0$  it must be for the WKB approximate solutions to be valid.

If  $P(x)$  vanishes in the way given by (7.14), we say that  $P(x)$  has an  $n^{\text{th}}$ -order zero at  $x_0$ . Not all zeroes of  $P(x)$  are of finite order, and an example of  $P(x)$  having a zero of infinite order is given by homework problem 6. The range of validity of the WKB solutions of homework problem 6 is given in homework problem 5b.

As we have mentioned, the WKB approximation is useful for describing the propagation of waves with very high frequencies or, equivalently, very small wavelengths. As an example, consider the problem of determining the shadow cast on a wall by a point light source in front of a screen. To obtain the exact solution of this problem, one solves the wave equation and make the solution satisfy the boundary conditions imposed by the presence of the screen, which is a difficult boundary-value problem. On the other hand, the shadow on the wall is very accurately determined simply by drawing straight lines from the light source to the edges of the screen. This is because when the wavelength of light is very small compared to the dimensions of the screen, the WKB approximation can be used to justify the results obtained with the use of geometric optics. footnote As another example, we know that Newtonian mechanics is an approximation of quantum mechanics. However, the behavior of atoms obeying the rules of wave mechanics is drastically different from that of particles obeying the rules of Newtonian mechanics. How does one reconcile these two sets of rules? The answer again lies in the WKB approximation, in which the Schrodinger equation is reduced to the Hamilton-Jacobi equation satisfied by the classical action of Newtonian mechanics.

The WKB approximation can also be used to solve problems in which the functional behavior is

The WKB Approximation rapidly growing or rapidly decaying other than rapidly oscillatory, an example being the problems of boundary layer which we will discuss in Chapter 9. Consider the equation

$$y'' - \eta^2 y = 0. \quad (7.16)$$

The WKB solutions are given by

$$y_{\text{WKB}}^{\pm}(x) = \frac{1}{\sqrt{\eta(x)}} e^{\pm \int \eta(x) dx}. \quad (7.17)$$

These solutions are good approximations of the solutions of (7.16) if

$$\left| \frac{d}{dx} \frac{1}{\eta(x)} \right| \ll 1. \quad (7.18)$$

The counterpart of (7.5) is

$$\eta(x) = \lambda N(x), \quad (7.19)$$

where  $\lambda \gg 1$  and  $N(x)$  is of order unity. As before, if  $\eta$  is in the form (7.19), the inequality (7.18) is always satisfied unless  $x$  is near a zero of  $N(x)$ .

The WKB approximation can be justified under other conditions. For example, it is easy to verify that (7.4) is satisfied if  $p(x)$  is of the form

$$p(x) = P(\epsilon x), \epsilon \ll 1, \quad (7.20)$$

where we have a small parameter  $\epsilon$  rather than a large parameter  $\lambda$ . If we make the change of variable

$$X = \epsilon x,$$

then (7.1) becomes

$$\frac{d^2 y}{dX^2} + \lambda^2 P^2(X) y = 0,$$

where the large parameter  $\lambda = \epsilon^{-1}$  appears.

Sometimes the large parameter  $\lambda$  is not explicitly exhibited. As an example, consider the problem of solving the Airy equation  $y'' - xy = 0$  when  $x$  is very large. In this problem,  $x$  inherently contains a large parameter. Indeed, let  $x$  be of the order of  $\Lambda$ , with  $\Lambda \gg 1$ . We may put

$$x \equiv \Lambda X,$$

where  $X$  is of the order of unity. Then the Airy equation is

$$\left( \frac{d^2}{dX^2} - \Lambda^3 X \right) y = 0.$$

Comparing with (7.16), we have, if  $X$  is positive,

$$\eta(X) = \lambda X^{1/2},$$

where  $\lambda \equiv \Lambda^{3/2}$  is the large parameter.

We also note that the integral  $\int p dx$  is dimensionless and hence does not change with a change of the scale of the independent variable. In the example of the Airy equation, we have

$$\int \lambda X^{1/2} dX = \int x^{1/2} dx.$$

Indeed, since the dimension of the wavelength is that of length, same as that of  $x$ , the leftside of (7.4) is dimensionless. Thus (7.4) can be expressed either with the variable  $X$  or with the variable  $x$ , as

$$\frac{d}{dX} \frac{1}{\lambda X^{1/2}} = \frac{d}{dx} \frac{1}{x^{1/2}}.$$

## B Solutions Near an Irregular Singular Point

We will now apply the WKB method to obtain the asymptotic solutions near an irregular singular point of a second-order linear homogeneous equation. While we have already given a method in the preceding section to obtain these solutions, it applies only when the rank of the singular point is an integer. In addition, the use of the WKB method makes it very easy to obtain the leading terms of the asymptotic series.

To start, we consider the leading asymptotic terms for the solutions of the equation

$$y'' + xy = 0. \quad (7.21)$$

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For  $x < 0$ , we have, comparing with (7.16),

$$\eta = (-x)^{1/2}.$$

Thus the WKB solutions are

$$|x|^{-1/4} e^{\pm 2|x|^{3/2}/3}. \quad (7.22)$$

We conclude immediately that, when  $x$  is large and negative, one of these solutions is an exponentially increasing function of  $x$  and the other is an exponentially decreasing function of  $x$ .

When  $x$  is positive, we have, comparing with (7.1),

$$p = x^{1/2}.$$

Thus the WKB solutions are

$$x^{-1/4} e^{\pm i2x^{3/2}/3}, \quad (7.23)$$

both being oscillatory functions of  $x$ .

## C Higher-order WKB Approximation

We shall in this section find the higher-order terms of the WKB approximation.

For this purpose let us return to eq. (7.11). Since this equation has a small parameter  $\epsilon$  and is linear, it is straightforward to use it to derive successive corrections to the WKB approximations. We put

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots, \quad (7.38)$$

where  $v_n$ ,  $n = 0, 1, \dots$ , are independent of  $\epsilon$ . The series in (7.38) is called a perturbation series which is expected to be useful when  $\epsilon$  is small. We substitute (7.38) into (7.11) and get

$$\begin{aligned} (v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots)' + \frac{P'}{2P} (v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots) \\ = \pm \frac{i\epsilon}{2P} (v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots)'' \end{aligned} \quad (7.39)$$

In the lowest-order approximation we set  $\epsilon$  in (7.39) to zero and get

$$v_0' + \frac{P'}{2P} v_0 = 0.$$

This equation gives

$$v_0(x) = \frac{1}{\sqrt{P(x)}},$$

which is, aside from an immaterial constant multiple, (7.13).

Setting to zero the sum of terms in (7.39) which are proportional to  $\epsilon$ , we get

$$v_1' + \frac{P'(x)}{2P(x)} v_1 = \pm \frac{i}{2P(x)} \left( \frac{1}{\sqrt{P(x)}} \right)'' \quad (7.40)$$

Solving this first-order linear equation, we find that

$$v_1(x) = \pm \frac{i}{2\sqrt{P(x)}} \int \frac{1}{\sqrt{P(t)}} \left( \frac{1}{\sqrt{P(t)}} \right)'' dt. \quad \#$$

Now we are ready to give a justification of the WKB method, which is approximating the solution of (7.1) by truncating the series of (7.38). Strictly speaking, truncating a series is justified if we succeed in proving that the sum of terms neglected is much less than the sum of terms kept. But proving this is sometimes difficult to do. We shall be content with proving that the  $(n+1)^{th}$  term in the series is much less than the  $n^{th}$  term if  $\epsilon$  is sufficiently small. Thus we will accept the WKB solutions (7.5) are good first-order approximations if

$$|\epsilon v_1| \ll |v_0|. \quad (7.42)$$

Since  $\epsilon$  is small, (7.42) is satisfied provided that  $v_1(x)/v_0(x)$  does not blow up, which is true unless  $P(x)$  happens to vanish.

If  $P(x)$  vanishes at  $x_0$ , the differential equation (7.1) is said to have a turning point at  $x_0$ . At a turning point of the differential equation, both  $v_0(x)$  and  $v_1(x)$  blow up, and the WKB approximation fails.

How far away from the turning point it must be in order for the WKB approximation to work? If,

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when  $x$  is near  $x_0$ ,  $P(x)$  goes to zero like  $(x - x_0)^n$ , then  $v_1(x)$  blows up like

$$(x - x_0)^{-1-3n/2}, \quad (7.43)$$

while  $v_0(x)$  blows up like  $(x - x_0)^{-n/2}$ . Thus (7.42) requires

$$|x - x_0| \gg \frac{1}{\lambda^{1/(1+n)}}. \quad (7.44)$$

Aside from a multiplicative constant, (7.44) is the same condition as (7.15).

We may find all higher-order terms of the solution from (7.39). This is done by gathering all the terms in (7.39) proportional to  $\epsilon^m$  and setting the sum to zero. We get

$$v'_m + \frac{P'}{2P} v_m = \pm \frac{i}{2P} v''_{m-1}. \quad (7.45)$$

Thus

$$v_m(x) = \pm \frac{i}{2\sqrt{P(x)}} \int \frac{dx}{\sqrt{P(x)}} \frac{d^2}{dx^2} v_{m-1}(x). \quad (7.46)$$

From (7.46), we obtain the  $m^{\text{th}}$ -order term of the perturbation series once the  $(m - 1)^{\text{th}}$ -order term of the perturbation series been found. Thus we obtain all  $v_m$  by successive iteration.

If  $P(x)$  has no zero, all  $v_m$  are finite. When  $\epsilon$  is sufficiently small, we have

$$|\epsilon v_m| \ll |v_{m-1}|. \quad (7.47)$$

Thus the WKB approximation is justified to higher orders. Here we like to give the reader a reminder: the WKB approximation has been justified to higher orders only if  $p(x)$  is of the form (7.5), or  $\eta(x)$  is of the form of (7.19), and neither of them vanish. I feel obligated to say it as I have seen the WKB approximations being too liberally applied.

We may show that, if  $P(x)$  vanishes at  $x_0$ , and is given by (7.14) near  $x_0$ , the condition (7.47) is satisfied provided that  $x$  is sufficiently far away from  $x_0$  so that (7.15) is satisfied. (See homework problem 3.)

Similarly, to obtain successive approximations to the WKB solutions of (7.16), we put

$$y \equiv e^{\pm \int \eta(x) dx} v. \quad (7.48)$$

Then we have

$$v' + \frac{\eta'}{2\eta} v = \mp \frac{1}{2\eta} v''. \quad (7.49)$$

We may use (7.49) to obtain successive approximations of the WKB solutions.

Homework due next Monday (Oct 18, 04) :

Chapter 7, 4a, 5b, 7b.