Problem Set # 6, 18.305. MIT (Fall 2005)

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Special Problem is: Traveling waves . . . .

Please hand in the special problem and the regular problems in two SEPARATE parts, each batch stapled together. Your name must be clearly typed in the front page of each batch.

Due date is: Monday December 12.

1 Periodic solution to Duffing's equation (statement).

All the solutions to the equation (Duffing's equation)

$$\frac{d^2\, y}{dt^2} + y + 2\,\epsilon\, y^3 = 0, \quad \text{where} \quad 0 < \epsilon \ll 1, \tag{1.1}$$

are periodic in time **(can you show this?)**. Consider now the initial value problem for equation (1.1), given by

$$y(0) = 1 \quad \text{and} \quad \frac{dy}{dt}(0) = 0. \tag{1.2}$$

Part (a) Show that the solution to (1.1 – 1.2) is an even function of t.

Part (b) If you attempt to expand the solution to (1.1 – 1.2) using a regular expansion of the form $y \sim \sum_{n=0}^{\infty} \epsilon^n \, y_n(t)$, you will find that the expansion breaks down for $t \gg 1$, and requires the condition $0 \leq t \ll \epsilon^{-1}$ for validity. In particular, y_1 will have a term proportional to $t \sin t$, y_2 a term proportional to $t^2 \cos t$, and so on.

The reason for the breakdown of the regular expansion above, is that the solution to (1.1 – 1.2) is periodic, but its period is not 2π, but only approximately 2π. Hence expand the solution as follows

$$y \sim \sum_{n=0}^{\infty} \epsilon^n \, y_n(\tau), \quad \text{where} \quad \tau = \omega t, \quad \omega = 1 + \epsilon\, \omega_1 + \epsilon^2\, \omega_2 + \dots \tag{1.3}$$

and the ω_n's are some constants. Of course, you do not know a priori what these constants are, but you should be able to determine them by requiring that each of the y_n's be a 2π periodic function of τ. Namely: when you plug the expansion into the equation, and collect equal powers of ϵ, you will find that at $O(\epsilon^n)$ you have to solve an equation of the form $\ddot{y_n} + y_n = F_n$, where $F_n(\tau)$ is some forcing term (and the dots indicate derivation with respect to τ). The constant ω_n will appear in F_n, and you will find that y_n is periodic if and only if ω_n takes a particular value.

Carry the expansion in (1.3) at least as far as determining y_1 and ω_1.

HINT: Change variables $t \to \tau$ before substituting the expansion for y into the equation.

NOTE: the approach in (1.3) is a particular case of the Poincaré - Linstead method.

Part (c) The solution to (1.1 – 1.2) can be written exactly in terms of quadratures. In particular, you can obtain an integral expression that gives the period T as a function of ϵ. Namely, an equation of the form

$$T = \int_a^b f(y,\, \epsilon)\, dy, \tag{1.4}$$

for some function f and some integration limits $a < b$. **Find this formula for the period.**

Part (d) Find the first two terms in a small ϵ expansion for the **period T, as given by equa-**

tion (1.4). Namely, **show that (1.4)** $\Longrightarrow$

$$T = T_0 + \epsilon T_1 + O(\epsilon^2), \tag{1.5}$$

for some **constants** T_0 **and** T_1 **that you should compute.**

Part (e) The expansion in **part b** gives a prediction for the period. Namely: $T = 2\pi/\omega$. **Show that this prediction agrees with your result in part d.**

2 Traveling waves (statement).

For linear and homogeneous in space wave problems, one can find traveling wave solutions expressed in terms of sine and cosine functions. For example, consider the simple equation

$$u_t - u_{xxx} = 0. \tag{2.1}$$

The traveling wave solutions of this equation are solutions of the form $u = y(\tau)$, where $\tau = k\,(x - s\,t)$ is the *phase*, y is periodic of period $2\,\pi$ in τ, s is the *phase speed*, and $k \neq 0$ is the *wavenumber* — related to the *wavelength* by $k = 2\,\pi/\lambda$. Substituting this form into the equation yields the ode:

$$s\,\frac{dy}{d\tau} + k^2\,\frac{d^3y}{d\tau^3} = 0. \tag{2.2}$$

The $2\,\pi$ periodic solutions to this equation are easy to find,[1] and are given by:

$$y = M + a\,\cos(\tau + \tau_0), \quad \text{where} \quad s = k^2, \tag{2.3}$$

and where M, $a > 0$ and τ_0 are constants (*wave mean, amplitude* and *phase shift*). **Notice that the phase speed** s **is not arbitrary: it is a function of the wavenumber** k.

For nonlinear problems we can also look for traveling wave solutions. An **important change is that the phase speed in these cases is a function not only of the wavenumber, but of the wave amplitude and mean as well.** For example, calculate the traveling waves for the equation

$$u_t + 3\,\epsilon\,u^2\,u_x - u_{xxx} = 0, \quad \text{where} \quad 0 < \epsilon \ll 1. \tag{2.4}$$

[1] Since it is a constant coefficients linear ode.

Do so by expanding

$$u = y(\tau) \sim \sum_{n=0}^{\infty} \epsilon^n \, y_n(\tau) \quad \text{and} \quad s = s_0(k) + \epsilon \, s_1(k,\, a,\, M) + \dots, \tag{2.5}$$

where the s_n's are functions to be found. **Calculate up to and including y_1 and s_1.**

HINTs: clearly y_0 and s_0 will be given by (2.3). The requirement that each of the functions y_n must be a 2π periodic function of τ will then determine the higher order s_n's. It is convenient to fix the phase shift by setting $\tau = 0$ to be a maximum of the wave, and to set the amplitude and the mean a-priory. That is: **Impose the conditions**

$$\boxed{\mathsf{Mean}(y) = M, \quad y(0) = a + M, \quad \text{and} \quad \frac{dy}{d\tau}(0) = 0, \quad \text{where } a > 0.} \tag{2.6}$$

Thus $\mathsf{Mean}(y_n) = M \, \delta_{0n}$, $y_n(0) = (M + a) \, \delta_{0n}$, and $\dot{y_n}(0) = 0$ — where δ_{ij} is Kronecker's symbol.

3 Problem # 1 in boundary layers (statement).

Find $y = y(x)$ approximately for the following boundary-value problems, where $\boxed{0 < \epsilon \ll 1.}$ In each case compute the leading order terms in each of the various regions that arise — including any boundary layers. No undetermined constants should be left.

Part (a) [20 points].

$$\epsilon \, y'' + (1 + x)^3 \, y' + 2 \, y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 0 \text{ and } y(1) = e^{1/4}. \tag{3.1}$$

HINT: Use dominant balance, and scaling of the independent variable, to find the equation(s) that must hold in any possible layer — where the solution varies on a short space scale. Then, from the properties of the solutions to these equations (growth or decay), determine where a layer may arise.

Part (b) [20 points].

$$\epsilon \, y'' - (1 + x)^3 \, y' + 2 \, y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 0 \text{ and } y(1) = 1. \tag{3.2}$$

HINT: Note that the change in the sign of the first derivative term will have an important effect on the boundary layer equations.

Part (c) [30 points].

$$\epsilon\, y'' - 2\,\sin x\; y' + \cos x\; y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 2 \text{ and } y(1) = \epsilon^{-1/4}. \tag{3.3}$$

HINT: Show that there is a boundary layer at $x = 0$ and another boundary layer at $x = 1$. Describe their widths, and derive approximate formulas for the slowly (away from the layers) and rapidly varying (in the layers) solutions. Note that the boundary layer at $x = 1$ is not very different in character from the boundary layers in the other two parts, but the boundary layer at $x = 0$ has a different character (because $\sin x$ vanishes there). You will need to use parabolic cylinder functions to deal with the behavior near $x = 0$.

THE END.