Problem Set # 5, 18.305. MIT (Fall 2005)

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Special Problem is: Pendulum ....

Please hand in the special problem and the regular problems in two SEPARATE parts, each batch stapled together. Your name must be clearly typed in the front page of each batch.

Due date is: Wednesday November 23.

1 Some asymptotic expansions (statement).

[Part (a)]. Let $y = y(x)$ have an asymptotic expansion of the form

$$y \sim \sum_{n=1}^{\infty} a_n\, x^n \sim a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \quad \text{for } 0 \leq x \ll 1, \text{ where } a_1 > 0. \tag{1.1}$$

Then we can write

$$x \sim \sum_{n=1}^{\infty} b_n\, y^n \sim b_1 y + b_2 y^2 + b_3 y^3 + \ldots, \quad \text{for } 0 \leq y \ll 1, \text{ where } b_1 > 0. \tag{1.2}$$

$\boxed{\textbf{FIND } b_1, b_2, \textbf{ and } b_3.}$

[**Part (b)**]. Let $y = y(x)$ have an asymptotic expansion of the form $y \sim \sum_{n=1}^{\infty} a_n\, x^{n/3}$, for $0 \le x \ll 1$, where $a_1 > 0$. **Modify (1.2) appropriately, and calculate the first two terms in an expansion for x in terms of y, for $0 \le y \ll 1$.**

[**Part (c)**]. Let $y = x - x^2 \ln x - \delta_1$ for $0 < x \ll 1$, where $\delta_1 = O(x^2)$. **Solve for x as a function of y, for $0 < y \ll 1$, with as many terms as possible — including an estimate for the error term in your solution.**

[**Part (d)**]. Find the first three terms in an expansion (valid for $0 < x \ll 1$) for the solutions of

$$\cos y + x - 1 + x^2 = 0, \tag{1.3}$$

where y is small.

2 Cole-Hopf transformation (statement).

Consider the equation for $u = u(x,t)$ (known as **Burgers'** equation)

$$u_t + 2uu_x = u_{xx}. \tag{2.1}$$

Introduce now $v = v(x,t)$ by $u = v_x$. **Show that one can choose v so that:**

$$v_t + v_x^2 = v_{xx}. \tag{2.2}$$

Let now $\Phi = e^{-v}$ (so that $v = -\ln \Phi$). **Show that Φ satisfies the heat equation:**

$$\Phi_t = \Phi_{xx}. \tag{2.3}$$

The net effect of all this is that the nonlinear equation (2.1) is linearized by the transformation $u = -\Phi_x/\Phi$. This transformation is known as the **Cole-Hopf transformation.**

3 Pendulum Small Amplitude Expansion (statement).

The (dimension-less) equation for a pendulum is

$$\frac{d^2y}{dt^2} + \sin y = 0. \tag{3.1}$$

Consider now a small amplitude expansion $y \sim \sum_{n=1}^{\infty} \epsilon^n \, y_n(t)$, for the solution with initial conditions $y(0) = 0$ and $\frac{dy}{dt}(0) = \epsilon$ — where $0 < \epsilon \ll 1$. Calculate a few terms in the expansion, and answer the following questions:

(a) Is the expansion valid for all $0 < t < \infty$? If no, how large can t be?

(b) If the answer to part (a) is no, can you explain the reason why a regular expansion does not work for this problem?

THE END.