Problem Set # 4, 18.305. MIT (Fall 2005)

D. Margetis and R. Rosales (MIT, Math. Dept., Cambridge, MA 02139).

November 05, 2005.

Course TA: Nikos Savva, MIT, Dept. of Mathematics, Cambridge, MA 02139.

Contents

Special Problem is: Klein Gordon . . . .

Please hand in the special problem and the regular problems in two SEPARATE parts, each batch stapled together. Your name must be clearly typed in the front page of each batch.

Due date is: Monday November 14.

1 Linear KdV equation asymptotics (statement).

The deviations u from a flat surface produced by a very small amplitude, uni-directional, long gravity wave disturbance in a water channel are described by the linear Korteweg de-Vries (KdV) equation. This equation can be written in the (dimension-less) form

$$u_t - \frac{1}{3}\,\epsilon^2\,u_{xxx} = 0, \quad \text{where} \quad 0 < \epsilon = h/\lambda \ll 1, \tag{1.1}$$

x is the length coordinate along the channel — in a frame moving at the infinite wavelength wave-speed $\sqrt{gh}$, λ is the wavelength, h is the channel depth, and g is the acceleration of gravity.

Given initial conditions for u, the solution to equation (1.1) above can be written in terms of Fourier Transforms, as follows:

$$u = \int_{-\infty}^{\infty} \exp\left\{ i \left(k\,x - \frac{\epsilon^2}{3}\,k^3\,t \right) \right\} g(k)\,dk, \tag{1.2}$$

where

$$g(k) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} u(x,0)\,\exp\left(-\,i\,k\,x\right)\,dx. \tag{1.3}$$

Consider now the situation where the initial data are localized in a very small region, of size ϵ. Namely, assume that

$$u(x,0) = U\left(\frac{x}{\epsilon}\right), \tag{1.4}$$

where $U = U(\zeta)$ decays very fast as $|\zeta| \to \infty$.

Part (a): Show that the Fourier Transform g — as given by the formula in (1.3) — for the initial data in (1.4), has the form $\boxed{g = \epsilon\,G\,(\epsilon\,k)}$ — for some function G.

Part (b): Use the result in part (a) to write the solution to the linear KdV equation in the form

$$u = \int_{-\infty}^{\infty} \exp\left\{ \frac{i}{\epsilon}\,\theta \right\} G\,d\kappa, \tag{1.5}$$

for appropriately selected variables, where $0 < \epsilon \ll 1$ shows up in the solution only in the exponent (as indicated in the formula above). **You need equation (1.5) to do parts (c-e) below.**

Part (c): Calculate the leading order behavior of the solution for fixed $x > 0$ and $t > 0$, as $\epsilon \downarrow 0$.

Part (d): Show that, for fixed $x < 0$ and $t > 0$, the solution vanishes to all orders in ϵ, as $\epsilon \downarrow 0$.

Part (e): Calculate the leading order behavior of the solution for fixed $t > 0$ and $x \approx 0$, as $\epsilon \downarrow 0$.

Hints for parts c-e: *Stationary phase, integration by parts, and Airy.*

2 An eigenvalue problem (statement).

Let $0 < \epsilon \ll 1$, and consider the eigenvalue problem

$$-\epsilon^2 y'' + V(x)y = Ey, \quad -\infty < x < \infty, \tag{2.1}$$

where V is given by: $V(x) = 1$ for $|x| > 1$ and $V(x) = 0$ for $|x| < 1$. The solutions must vanish as $x \to \pm\infty$ and must be continuous with continuous first derivatives.

Calculate the eigenfunctions and eigenvalues for this problem exactly. Compare the eigenvalues thus obtained with the answer provided by equation (10.5.6) in Bender & Orszag's book. EXPLAIN any discrepancies.

HINT: The eigenvalues exist only in the range $0 \le E \le 1$ **(why? – there is an easy argument!)** For $0 < E < 1$ the transformation

$$1 - 2E = \cos\phi \quad \text{and} \quad 2\sqrt{(1-E)\,E} = \sin\phi, \quad \text{with} \quad 0 < \phi < \pi,$$

is useful when writing the equation that the eigenvalues must satisfy. The cases $E = 0$ and $E = 1$ must be treated separately.

3 Connection formulas (statement).

Let $0 < \epsilon \ll 1$, and consider the ODE problem

$$\epsilon^2 y'' = Q(x)y, \tag{3.1}$$

where $Q(x) > 0$ for $x > 0$, and $Q(x) < 0$ for $x < 0$. However, Q does not have a simple zero at $x = 0$. Instead, assume that Q has a jump discontinuity at $x = 0$ — with Q smooth on each side, with left and right limits for it and all its derivatives.

Calculate the connection formulas across the origin, for the WKB approximations for the problem stated above.

4 Klein Gordon Reflection & Transm. Coeff. (statement).

Consider a string under tension, attached to a (very stiff) elastic bed, with the stiffness of the elastic bed varying along the string length. The small deviations u from equilibrium (straight string) then satisfy the Klein Gordon equation, which can be written in the dimension-less form

$$u_{tt} - u_{xx} + \frac{1}{\epsilon^2} V(x) u = 0, \quad \text{where} \quad 0 < \epsilon \ll 1. \tag{4.1}$$

Assume now that V is a continuous even function, vanishes outside some finite interval $|x| < a$, and is positive inside it. In fact, assume that $dV/dx > 0$ for $-a < x < 0$, with a single maximum at the origin, where $V(0) = 1$. In this case the solutions outside of the interval $|x| < a$ must have the simple form: $\boxed{u = f(x - t) + g(x + t)}$ — for some functions f and g. In other words, they are the superposition of two steady waves with unit speed: one moving to the right and the other moving to the left.

In particular, we now look at the situation where a single frequency wave is launched from $x = -\infty$, and we want to compute how much is reflected and how much is transmitted. This problem can be formulated mathematically as follows: Find the solution defined by the following properties

(A) For $x < -a$ .... $\boxed{u = \exp\left\{i\,\frac{\omega}{\epsilon}\,(t - x)\right\} + R\,\exp\left\{i\,\frac{\omega}{\epsilon}\,(t + x)\right\},}$

(B) For $x > +a$ .... $\boxed{u = T\,\exp\left\{i\,\frac{\omega}{\epsilon}\,(t - x)\right\},}$

where ω, R, and T are constants, with ω given (and real) and the other two are to be determined.

(a) Use WKB techniques to produce approximate expressions for R and T.

(b) When $V = 1 - |x|$ for $|x| < 1$, give explicit (approximate) formulas for R and T.

EXTRA CREDIT (15 points): In case (b) above the problem can be solved exactly, and the WKB approximation can be checked. Do this. **Important:** *The extra credit applies ONLY if you do this right. You can, and will, get NEGATIVE credit if you botch this up, even minimally. Hence: hand this part in ONLY if you are 100% certain of your answer.*

THE END.