Problem Set # 3, 18.305. MIT (Fall 2005)

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Special Problem is: 3-rd order equation WKB.

Due date is: Monday October 31.

1 Airy functions integrals (statement).

Consider three paths in the complex plane with the following properties:

- Γ_1 goes from $|z| = \infty$ along the radial line $\arg(z) = (5/6)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (1/6)\pi$.
- Γ_2 goes from $|z| = \infty$ along the radial line $\arg(z) = (3/2)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (1/6)\pi$.
- Γ_3 goes from $|z| = \infty$ along the radial line $\arg(z) = (5/6)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (3/2)\pi$.

In the lectures it was shown that

$$Ai(x) = \frac{1}{2\pi} \int_{\Gamma_1} e^{i(xz + \frac{1}{3} z^3)}\, dz \quad \text{and} \quad Bi(x) = \frac{1}{2\pi i} \int_{\Gamma_2} e^{i(xz + \frac{1}{3} z^3)}\, dz + \text{c.c.}. \tag{1.1}$$

are both solutions of the Airy equation $y'' = xy$, where c.c. denotes the complex conjugate.

Show that the above are equivalent to:

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos\left(xt + \frac{1}{3} t^3\right)\, dt, \tag{1.2}$$

and

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \sin\left(xt + \frac{1}{3} t^3\right)\, dt \quad + \quad \frac{1}{\pi} \int_0^\infty \exp\left(xt - \frac{1}{3} t^3\right)\, dt. \tag{1.3}$$

2 Airy functions expansions (statement).

In the lectures we showed that the exponentially decaying solutions for the Airy equation $\epsilon^2\, y'' = xy$ (for $x > 0$ and $0 < \epsilon \ll 1$), had an asymptotic expansion of the form

$$y \sim x^{-1/4} \left(\sum_0^\infty a_n\, (\epsilon/\zeta)^n \right)\, e^{-\zeta/\epsilon} \quad \text{where} \quad \zeta = \frac{2}{3}\, x^{3/2}. \tag{2.1}$$

Part (a) Calculate the coefficients a_n assuming that $a_0 = 1$.

For $x < 0$ the solutions admit asymptotic expansions of the form

$$y \sim |x|^{-1/4} \left(\sum_0^\infty d_n\, (\epsilon/\eta)^n \right)\, e^{i\eta/\epsilon} \tag{2.2}$$

and its complex conjugate, where $\eta = \dfrac{2}{3}\, |x|^{3/2}$.

Part (b) Calculate the coefficients d_n assuming that $d_0 = 1$.

3 Generic 2nd order equation WKB (statement).

Let $0 < \epsilon \ll 1$, and consider the second order ODE

$$\epsilon^2\, y'' + \epsilon a\, y' + hy = 0, \tag{3.1}$$

where $a = a(x)$ and $h = h(x)$ are some given functions.

(a) Use a WKB-like approach to get asymptotic expansions for the solutions of this equation.

(b) Where do you expect the expansions obtained in the prior step to break down? Explain.

Note: The expansions will have the form $y \sim A\,e^S$, where S is appropriately selected and A has an expansion of the form $A \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \dots$. You should write the equations satisfied by the A_n.

4 WKB expansion for 3-rd order equation (statement).

Let $0 < \epsilon \ll 1$, and consider the 3-rd order ODE

$$\epsilon^3 \frac{d^3 y}{dx^3} + V y = 0, \tag{4.1}$$

where $V = V(x)$ is some given function.

(a) Use a WKB-like approach to get asymptotic expansions for the solutions of this equation.

(b) Where do you expect the expansions obtained in the prior step to break down? Explain.

(c) Consider the case $V = -x$, and $x > 0$. What is the condition for validity of these expansions?

Note: The expansions will have the form $y \sim A\,e^S$, where S is appropriately selected and A has an expansion of the form $A \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \dots$. For part (a) you should provide expressions for S and (at least) the leading order term A_0. For part (c) the A_n will be powers of x, and your answer should take the form that the expansions are valid provided $\epsilon^\mu \ll x$, for some μ.

THE END.