Answers to Problem Set Number 6.
18.305 — MIT (Fall 2005).

D. Margetis and R. Rosales (MIT, Math. Dept., Cambridge, MA 02139).

December 12, 2005.

Course TA: Nikos Savva, MIT, Dept. of Mathematics, Cambridge, MA 02139.

Contents

1 Periodic solution to Duffing's equation (statement).

All the solutions to the equation (Duffing's equation)

$$\frac{d^2 y}{dt^2} + y + 2\,\epsilon\, y^3 = 0, \quad \text{where} \quad 0 < \epsilon \ll 1, \tag{1.1}$$

are periodic in time **(can you show this?)**. Consider now the initial value problem for equation (1.1), given by

$$y(0) = 1 \quad \text{and} \quad \frac{dy}{dt}(0) = 0. \tag{1.2}$$

Part (a) Show that the solution to (1.1 – 1.2) is an even function of t.

Part (b) If you attempt to expand the solution to (1.1 – 1.2) using a regular expansion of the form $y \sim \sum_{n=0}^{\infty} \epsilon^n\, y_n(t)$, you will find that the expansion breaks down for $t \gg 1$, and requires the condition $0 \le t \ll \epsilon^{-1}$ for validity. In particular, y_1 will have a term proportional to $t\,\sin t$, y_2 a term proportional to $t^2\,\cos t$, and so on.

The reason for the breakdown of the regular expansion above, is that the solution to (1.1 – 1.2) is periodic, but its period is not 2π, but only approximately 2π. Hence expand the solution as follows

$$y \sim \sum_{n=0}^{\infty} \epsilon^n \, y_n(\tau), \quad \text{where} \quad \tau = \omega t, \quad \omega = 1 + \epsilon\,\omega_1 + \epsilon^2\,\omega_2 + \dots \tag{1.3}$$

and the ω_n's are some constants. Of course, you do not know a priori what these constants are, but you should be able to determine them by requiring that each of the y_n's be a 2π periodic function of τ. Namely: when you plug the expansion into the equation, and collect equal powers of ϵ, you will find that at $O(\epsilon^n)$ you have to solve an equation of the form $\ddot{y_n} + y_n = F_n$, where $F_n(\tau)$ is some forcing term (and the dots indicate derivation with respect to τ). The constant ω_n will appear in F_n, and you will find that y_n is periodic if and only if ω_n takes a particular value.

Carry the expansion in (1.3) at least as far as determining y_1 and ω_1.

HINT: Change variables $t \to \tau$ before substituting the expansion for y into the equation.

NOTE: the approach in (1.3) is a particular case of the Poincaré - Linstead method.

Part (c) The solution to (1.1 – 1.2) can be written exactly in terms of quadratures. In particular, you can obtain an integral expression that gives the period T as a function of ϵ. Namely, an equation of the form

$$T = \int_a^b f(y,\,\epsilon)\,dy, \tag{1.4}$$

for some function f and some integration limits $a < b$. **Find this formula for the period.**

Part (d) Find the first two terms in a small ϵ expansion for the **period T, as given by equation (1.4).** Namely, **show that (1.4) $\Longrightarrow$**

$$T = T_0 + \epsilon T_1 + O(\epsilon^2), \tag{1.5}$$

for some **constants T_0 and T_1 that you should compute.**

Part (e) The expansion in **part b** gives a prediction for the period. Namely: $T = 2\pi/\omega$. **Show that this prediction agrees with your result in part d.**

Solution to the periodic . . . Duffing's equation problem.

Multiply equation (1.1) by $2\,\dfrac{dy}{dt}$, an integrate. This yields:

$$\left(\frac{d\,y}{dt}\right)^2 + y^2 + \epsilon\, y^4 = E, \quad \text{where} \quad E \text{ is a constant.} \tag{1.6}$$

Thus, in the **phase plane** $(y,\, z)$ — where $z = \dfrac{dy}{dt}$ — the solutions are restricted to the level curves of $z^2 + y^2 + \epsilon\, y^4$, which are all closed simple curves. Hence **all the solutions are periodic.**

Solution to part (a) Both equation (1.1) and the initial values (1.2) are invariant under the transformation $t \to -t$. Thus, if $y = Y(t)$ is a solution to (1.1 – 1.2), $y = Y(-t)$ is also a solution. But the solution to an initial value problem is unique. It follows that the solution to (1.1 – 1.2) must be an even function of t.

Solution to part (b) First we transform variables $t \to \tau$. Thus:

$$\omega^2\, \frac{d^2\, y}{d\tau^2} + y + 2\,\epsilon\, y^3 = 0. \tag{1.7}$$

Now substitute (1.3) into (1.7), and collect equal powers of ϵ. This yields:

$O(\epsilon^0)$ $\ddot{y}_0 + y_0 = 0$, with $y_0(0) = 1$ and $\dot{y}_0(0) = 0 \implies$ . . . . $\boxed{y_0 = \cos(\tau).}$

$O(\epsilon^1)$ $\ddot{y}_1 + y_1 = -2\,\left(\omega_1\,\ddot{y}_0 + y_0^3\right) = 2\,(\omega_1 - 3/4)\,\cos t - (1/2)\,\cos 3t,$

with $y_1(0) = \dot{y}_0(0) = 0 \implies$ . . . . $\boxed{y_1 = \dfrac{1}{16}\,(\cos 3t - \cos t)\,,}$

where we **must choose** . . . . $\boxed{\omega_1 = 3/4}$

since otherwise y_1 would have a non-periodic term in it.

Solution to part (c) Using the initial values, we can evaluate $E = 1 + \epsilon$ in equation (1.6) for the problem in (1.1 – 1.2). Thus the solution satisfies

$$\frac{dy}{dt} = \pm\sqrt{(1 - y^2)\,\,(1 + \epsilon\,\,(1 + y^2))}. \tag{1.8}$$

It follows that the solution oscillates between $y = \pm 1$, with a half period the time it takes the solution to go from $y = -1$ to $y = 1$. Hence we can write:

$$T = 2\int_{-1}^{+1} \frac{dy}{\sqrt{(1 - y^2)\,\,(1 + \epsilon\,\,(1 + y^2))}} = 4\int_0^1 \frac{dy}{\sqrt{(1 - y^2)\,\,(1 + \epsilon\,\,(1 + y^2))}}. \tag{1.9}$$

$\boxed{\textbf{Solution to part (d)}}$ We use the binomial theorem to expand the factor involving ϵ in the integral in (1.9). This yields:

$$T = 4 \int_0^1 \frac{dy}{\sqrt{(1-y^2)}} - 2\,\epsilon \int_0^1 \frac{(1+y^2)}{\sqrt{(1-y^2)}}\,dy + O(\epsilon^2). \tag{1.10}$$

The substitution $y = \sin\theta$ transforms these integrals into easy ones. Evaluating them we then get:

$$T = 2\,\pi - \epsilon\,\frac{3}{2}\,\pi + O(\epsilon^2). \tag{1.11}$$

$\boxed{\textbf{Solution to part (e)}}$ The expansion in **part b** predicts a period:

$$T = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{1+\epsilon\,\frac{3}{4}+O(\epsilon^2)} = 2\,\pi\,\left(1-\epsilon\,\frac{3}{4}+O(\epsilon^2)\right) = 2\,\pi - \epsilon\,\frac{3}{2}\,\pi + O(\epsilon^2), \tag{1.12}$$

which obviously matches the exact period expansion in (1.11) of **part d**.

2 Traveling waves (statement).

For linear and homogeneous in space wave problems, one can find traveling wave solutions expressed in terms of sine and cosine functions. For example, consider the simple equation

$$u_t - u_{xxx} = 0. \tag{2.1}$$

The traveling wave solutions of this equation are solutions of the form $u = y(\tau)$, where $\tau = k\,(x - s\,t)$ is the *phase*, y is periodic of period $2\,\pi$ in τ, s is the *phase speed*, and $k \neq 0$ is the *wavenumber* — related to the *wavelength* by $k = 2\,\pi/\lambda$. Substituting this form into the equation yields the ode:

$$s\,\frac{dy}{d\tau} + k^2\,\frac{d^3y}{d\tau^3} = 0. \tag{2.2}$$

The $2\,\pi$ periodic solutions to this equation are easy to find,[1] and are given by:

$$y = M + a\,\cos(\tau+\tau_0), \quad \text{where} \quad s = k^2, \tag{2.3}$$

and where M, $a > 0$ and τ_0 are constants (*wave mean, amplitude* and *phase shift*). **Notice that the phase speed s is not arbitrary: it is a function of the wavenumber k.**

[1] Since it is a constant coefficients linear ode.

For nonlinear problems we can also look for traveling wave solutions. An **important change is that the phase speed in these cases is a function not only of the wavenumber, but of the wave amplitude and mean as well.** For example, calculate the traveling waves for the equation

$$u_t + 3\,\epsilon\, u^2\, u_x - u_{xxx} = 0, \quad \text{where} \quad 0 < \epsilon \ll 1. \tag{2.4}$$

Do so by expanding

$$u = y(\tau) \sim \sum_{n=0}^{\infty} \epsilon^n\, y_n(\tau) \quad \text{and} \quad s = s_0(k) + \epsilon\, s_1(k,\, a,\, M) + \dots, \tag{2.5}$$

where the s_n's are functions to be found. **Calculate up to and including y_1 and s_1.**

HINTs: clearly y_0 and s_0 will be given by (2.3). The requirement that each of the functions y_n must be a 2π periodic function of τ will then determine the higher order s_n's. It is convenient to fix the phase shift by setting $\tau = 0$ to be a maximum of the wave, and to set the amplitude and the mean a-priory. That is: **Impose the conditions**

$$\boxed{\mathsf{Mean}(y) = M, \quad y(0) = a + M, \quad \text{and} \quad \frac{dy}{d\tau}(0) = 0, \quad \text{where } a > 0.} \tag{2.6}$$

Thus $\mathsf{Mean}(y_n) = M\,\delta_{0n}$, $y_n(0) = (M+a)\,\delta_{0n}$, and $\dot{y}_n(0) = 0$ — where δ_{ij} is Kronecker's symbol.

Solution to the traveling waves problem.

Substituting $u = y(\tau)$ — where $\tau = k\,(x - s\,t)$ — into equation (2.4), we obtain the ode

$$s\,\frac{dy}{d\tau} + k^2\,\frac{d^3y}{d\tau^3} = \epsilon\,\frac{d}{d\tau}\left(y^3\right). \tag{2.7}$$

We **must find 2π-periodic solutions of this ode, satisfying (2.6).** Thus we substitute the expansion in (2.5) into this last equation, and solve order by order, as follows below.

(A) $O(1)$ terms. We obtain

$$s_0\,\frac{dy_0}{d\tau} + k^2\,\frac{d^3y_0}{d\tau^3} = 0. \tag{2.8}$$

The general solution to this problem has the form $y_0 = c_1 + c_2\,\cos(\sqrt{s_0}\,\tau/k) + c_3\,\sin(\sqrt{s_0}\,\tau/k)$, where the c_j's are constants. Hence, in order to satisfy 2π-periodicity and (2.6), we must take

$$y_0 = M + a\,\cos\tau, \quad \text{with} \quad s_0 = k^2. \tag{2.9}$$

(B) $O(\epsilon)$ **terms.** We obtain

$$s_0\,\frac{dy_1}{d\tau} + k^2\,\frac{d^3y_1}{d\tau^3} = \frac{d}{d\tau}\,\left(y_0^3\right) - s_1\,\frac{d}{d\tau}\,(y_0)\,. \tag{2.10}$$

That is

$$k^2\,\frac{d}{d\tau}\left(y_1 + \frac{d^2y_1}{d\tau^2}\right) = \frac{d}{d\tau}\left\{\left(3\,M^2\,a + \frac{3}{4}\,a^3 - s_1\,a\right)\,\cos\tau + \frac{3}{2}\,M\,a^2\,\cos 2\tau + \frac{1}{4}\,a^3\,\cos 3\tau\right\}. \tag{2.11}$$

Thus we **must select** .... $\boxed{s_1 = 3\,M^2 + \frac{3}{4}\,a^2}$
for the solution y_1 to this last equation to be periodic.

Then .... $\boxed{y_1 = \left(\frac{1}{2}\,M\,a^2\,k^{-2} + \frac{1}{32}\,a^3\,k^{-2}\right)\,\cos\tau - \frac{1}{2}\,M\,a^2\,k^{-2}\,\cos 2\tau - \frac{1}{32}\,a^3\,k^{-2}\,\cos 3\tau}$
satisfies the conditions in (2.6).

Notice that, in this example, s_1 is not a function of k. But at higher order in the expansion for the phase speed, one finds that s_n (for $n > 1$) depends on k.

3 Problem # 1 in boundary layers (statement).

Find $y = y(x)$ approximately for the following boundary-value problems, where $\boxed{0 < \epsilon \ll 1.}$ In each case compute the leading order terms in each of the various regions that arise — including any boundary layers. No undetermined constants should be left.

Part (a)

$$\epsilon\,y'' + (1+x)^3\,y' + 2\,y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 0 \ \text{ and } \ y(1) = e^{1/4}. \tag{3.1}$$

HINT: Use dominant balance, and scaling of the independent variable, to find the equation(s) that must hold in any possible layer — where the solution varies on a short space scale. Then, from the properties of the solutions to these equations (growth or decay), determine where a layer may arise.

Part (b)

$$\epsilon\,y'' - (1+x)^3\,y' + 2\,y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 0 \ \text{ and } \ y(1) = 1. \tag{3.2}$$

HINT: Note that the change in the sign of the first derivative term will have an important effect on the boundary layer equations.

Part (c)

$$\epsilon\, y'' - 2\,\sin x\, y' + \cos x\, y = 0, \qquad \text{for } 0 < x < 1, \quad \text{with} \quad y(0) = 2 \text{ and } y(1) = \epsilon^{-1/4}. \tag{3.3}$$

HINT: Show that there is a boundary layer at $x = 0$ and another boundary layer at $x = 1$. Describe their widths, and derive approximate formulas for the slowly (away from the layers) and rapidly varying (in the layers) solutions. Note that the boundary layer at $x = 1$ is not very different in character from the boundary layers in the other two parts, but the boundary layer at $x = 0$ has a different character (because $\sin x$ vanishes there). You will need to use parabolic cylinder functions to deal with the behavior near $x = 0$.

Solution to problem # 1 in boundary layers.

Part (a)

(a.1) A **regular expansion** $y \sim \sum \epsilon^n\, y_n(x)$ yields, at leading order $(1+x)^3\, y_0' + 2\, y_0 = 0$, with solution $\boxed{y_0 = c_1\, \exp\left\{(1+x)^{-2}\right\}}$ — where c_1 is a constant. This can satisfy, at most, one of the boundary conditions. Thus **a rapidly varying layer is needed somewhere.**

NOTE 1. Calculation of the higher order terms in this regular expansion shows no indication of breakdown anywhere for $0 \le x \le 1$. In other words $\epsilon\, y_{n+1}(x) \ll y_n(x)$ for all n and all $0 \le x \le 1$. Nevertheless, **this regular expansion is valid only for** $\boxed{\epsilon \ll x \le 1.}$ The reason is that terms proportional to $e^{-\zeta}$ — ζ as defined in **(a.3)** — are needed to approximate the solution near $x = 0$, which are missing in this expansion. In order to be able to neglect these terms, $x \gg \epsilon$ is needed.

(a.2) To search for the appropriate layer expansion, we begin by doing a **dominant balance analysis.** At a layer, the second derivative term must become important — hence the derivatives will be large. Because the first derivative term has a coefficient that is always positive, this term will be large in the layer[2] — thus it must be the one that balances the second derivative term. Furthermore,

[2]While the 3-rd term in the equation remains $O(1)$.

a balance between the first and second terms in equation (3.1) yields exponentially decreasing behavior. Hence: **there is a boundary layer, it occurs at the left end of the interval, and it entails a balance of the first two terms in equation (3.1).**

(a.3) Using the result of **(a.2)**, we introduce the layer variable $\boxed{\zeta = x/\epsilon}$ — so that the equation becomes

$$\frac{d^2 y}{d\zeta^2} + (1 + \epsilon\,\zeta)^3\,\frac{dy}{d\zeta} + 2\,\epsilon\,y = 0. \tag{3.4}$$

The leading order (set $\epsilon = 0$) solution for this equation has the form $y \sim c_2 + c_3\,e^{-\zeta}$, with c_1 and c_2 constants. This solution must satisfy the boundary condition at $x = 0$, hence $\boxed{y \sim c_2\left(1 - e^{-\zeta}\right).}$

NOTE 2. The higher orders in this boundary layer approximation have terms proportional to $c_2\,(\epsilon\,\zeta)^n$. Thus **the boundary layer expansion is valid for** $\boxed{0 \leq \epsilon\,\zeta = x \ll 1.}$

(a.4) From **(a.2)** we know that the regular expansion in **(a.1)** must satisfy the boundary condition on $x = 1$. Hence $c_1 = 1$. On the other hand, the regular expansion in **(a.1)**, and the boundary layer approximation in **(a.3)** must match for $\epsilon \ll x \ll 1$ (i.e. $1 \ll \zeta \ll 1/\epsilon$). Hence $c_2 = e$.

Putting it all together, we **get the following approximation for the solution**

For $\epsilon \ll x \leq 1$ .... $\boxed{y \sim \exp\left\{(1+x)^{-2}\right\}.}$

For $0 \leq x \ll 1$ .... $\boxed{y \sim e\left(1 - e^{-\zeta}\right).}$

Uniformly valid approximation .... $\boxed{y \sim \exp\left\{(1+x)^{-2}\right\} - \exp\left\{1 - \zeta\right\}.}$

Part (b)

The solution is very similar to the problem in **part (a)**. As in **(a.1)**, the need for a layer somewhere becomes apparent because a regular expansion $y \sim \sum \epsilon^n y_n(x)$ can only satisfy, at most, one boundary condition. Then the same type of argument as in **(a.2)** shows that: **there is a boundary layer, it occurs on the right end of the interval, and it entails a balance of the first two terms in equation (3.2).** Thus the solution of the problem proceeds as follows:

(b.1) Regular expansion $y \sim \sum \epsilon^n y_n(x)$, satisfying the boundary condition on the left $y(0) = 0$. It is easy to see that **this expansion vanishes to all orders.** As in **NOTE 1** — since terms proportional to $e^{8\zeta}$ (with ζ as in **(b.2)**) are missing from this expansion — we conclude that: **this regular expansion is valid for** $\boxed{\epsilon \ll 1 - x \leq 1,}$ **where y is smaller than any power of ϵ.**

(b.2) Right boundary layer expansion $y \sim \sum \epsilon^n y_n(\zeta)$ — where $\boxed{\zeta = \dfrac{x-1}{\epsilon}}$ — satisfying the boundary condition on the right $y(1) = 1$. In terms of ζ, the equation is

$$\frac{d^2 y}{d\zeta^2} - (2 + \epsilon\,\zeta)^3 \frac{dy}{d\zeta} + 2\,\epsilon\, y = 0. \tag{3.5}$$

Hence the general solution for $y_0(\zeta)$, satisfying the boundary condition for $\zeta = 0$, has the form $y_0 = c + (1-c)\,e^{8\zeta}$ — where c is a constant. Because the expansions in **(b.1)** and **(b.2)** have to match, we conclude that $c = 0$. Thus .... $\boxed{y \sim e^{8\zeta}.}$

NOTE 3. The higher orders in this boundary layer approximation have terms involving $(\epsilon\,\zeta^2)^n\, e^{8\zeta}$, $(\epsilon^2\,\zeta^3)^n\, e^{8\zeta}$, and $(\epsilon^3\,\zeta^4)^n\, e^{8\zeta}$. Thus **the boundary layer expansion is valid for** $\boxed{0 \leq \epsilon\,\zeta^2 \ll 1.}$ **Equivalently** .... $\boxed{0 \leq 1 - x \ll \sqrt{\epsilon}.}$

(b.3) WKB-like approximation. Finally, we point out that we can look for solutions of equation (3.2) of the form $y = A\, e^{S(x)/\epsilon}$, where $A \sim \sum \epsilon^n A_n(x)$. This then leads to the approximation

$$y \sim \left(8\,t^{-3} + O(\epsilon)\right)\, \exp\left\{\frac{t^4 - 16}{4\,\epsilon}\right\}, \quad \text{where} \quad t = 1 + x = 2 + \epsilon\,\zeta. \tag{3.6}$$

The results in **(b.2)** follow upon substituting $t = 2 + \epsilon\,\zeta$, and expanding appropriately for $\epsilon\,\zeta^2 \ll 1$. The results in **(b.1)** follow because this expression is exponentially small for $\epsilon \ll 1 - x \leq 1$, since then $\epsilon \ll 2 - t \leq 1 \quad \Longrightarrow \quad 16 - t^2 \gg \epsilon$.

Part (c)

(c.1) Regular expansion. Substituting $y \sim \sum \epsilon^n y_n(x)$ into the equation yields

$$2\,(\sin x)^{3/2} \left(\frac{y_n}{\sqrt{\sin x}}\right)' = 2\,\sin x\; y_n' - \cos x\; y_n = y_{n-1}'' \qquad \text{for } \; n > 1, \tag{3.7}$$

with $2\,\sin x\; y_0' - \cos x\; y_0 = 0$. Thus $y_0 = c_r\,\sqrt{\sin x} \;\Longrightarrow$ .... $\boxed{y \sim c_r\,\sqrt{\sin x},}$
where c_r is some constant.

Furthermore, it is easy to see that $y_n = O\left(x^{0.5 - 2\,n}\right)$ for $0 \leq x \ll 1$.
Then, from the condition $\epsilon\, y_{n+1} \ll y_n$, we conclude that .... **validity requires** $\boxed{x \gg \sqrt{\epsilon}.}$

Clearly this expansion cannot satisfy the boundary condition on the left. Below we will see that it can neither satisfy the boundary condition on the right. **Two boundary layers will be required, one on the left in terms of the variable** $\boxed{s = \sqrt{(2/\epsilon)}\, x,}$ **and another one on the right, in terms of the variable** $\boxed{\zeta = (x-1)/\epsilon.}$ Because terms involving exponentials of ζ are missing from this expansion, we have that .... **a further condition for validity is** $\boxed{1 - x \gg \epsilon.}$

(c.2) Layer Analysis. In any layer that arises, the second derivative term in equation (3.3) must become important (otherwise we have the situation in **(c.1)**). Thus the derivatives must be large (rapid variation). For $0 < x \leq 1$, $\sin x > 0$ and the two terms in (3.3) with derivatives will be large, balancing each other, and producing exponential growth. Hence a layer of this type can only appear on the right hand side of the interval of interest, and have width $O(\epsilon)$ — see **(c.5)**. This analysis fails near $x = 0$, because there $\sin x$ vanishes linearly, so that the second term in equation (3.3) ends up being of invariant size under scaling. Hence for the second derivative term to become important near $x = 0$ (and still have another term around to balance it), a layer of width $\sqrt{\epsilon}$ is needed — see **(c.3)**.

(c.3) Boundary Layer at $\boxed{x = 0.}$ The natural scaling to make the first two terms in equation (3.3) of the same size is $t = x/\sqrt{\epsilon}$. Then the equation becomes:

$$\frac{d^2 y}{d t^2} - 2t\left(1 - \frac{1}{6}\epsilon t^2 + O(\epsilon^2 t^4)\right)\frac{d y}{d t} + \left(1 - \frac{1}{2}\epsilon t^2 + O(\epsilon^2 t^4)\right) y = 0, \tag{3.8}$$

whose **leading order is a parabolic-cylinder equation** $\boxed{\ddot{y}_0 - 2t\,\dot{y}_0 + y = 0.}$ To solve this equation, let us first eliminate the first derivative term in the standard fashion: let $\boxed{y_0 = u\,\exp\left\{t^2/2\right\}.}$ Then $\boxed{\ddot{u} + (2 - t^2)\,u = 0.}$ Finally, $\boxed{s = \sqrt{2}\,t = \sqrt{(2/\epsilon)}\,x}$ yields

$$\frac{d^2 u}{d s^2} - \left(\frac{1}{4}s^2 - 1\right) u = 0, \tag{3.9}$$

which is a standard form for the parabolic-cylinder equation.[3] Only the solutions that decay as $s \to \infty$ are acceptable, hence $u = c_0\,U(-1, s)$, where c_0 is a constant to be determined by the boundary condition at $x = 0$. Since $U(-1, 0) = \sqrt{\pi}\,2^{1/4}/\Gamma(1/4)$, it follows that **the leading order**

[3] See chapter 19 in *Handbook of Mathematical Functions*, by Abramowitz and Stegun — published by Dover. $U(a, x)$ is defined in 19.3.1, the value $U(a, 0)$ is given in 19.3.5, and the asymptotic behavior is given in 19.8.1.

approximation for the solution in the left boundary layer is given by

$$y \sim 2\,\frac{\Gamma(1/4)}{2^{1/4}\sqrt{\pi}}\,U(-1,s)\;\exp\left\{\frac{1}{4}\,s^2\right\},\quad \text{where}\quad s=\sqrt{\frac{2}{\epsilon}}\,x. \tag{3.10}$$

(c.4) Matching of the left boundary layer in (c.3) and the regular expansion in (c.1). Since we have $U(-1,s) \sim \sqrt{s}\,e^{-s^2/4}$ for $s \gg 1$, the approximation in (3.10) yields

$$y \sim 2\,\frac{\Gamma(1/4)}{2^{1/4}\sqrt{\pi}}\,\sqrt{s} = 2\,\frac{\Gamma(1/4)}{\epsilon^{1/4}\sqrt{\pi}}\,\sqrt{x}.$$

This will match the expansion in **(c.1)** provided we take $c_r = 2\,\dfrac{\Gamma(1/4)}{\epsilon^{1/4}\sqrt{\pi}}$. Hence, **the leading order approximation for the solution, valid in the region $x \gg \sqrt{\epsilon}$ and $1 - x \gg \epsilon$, is given by**

$$y \sim 2\,\frac{\Gamma(1/4)}{\epsilon^{1/4}\sqrt{\pi}}\,\sqrt{\sin x}. \tag{3.11}$$

IMPORTANT: note the factor $\epsilon^{1/4}$ in the denominator. The solution is big!

(c.5) Boundary Layer at $\boxed{x = 1.}$ Introduce the variable $\zeta = (x-1)/\epsilon$. Then the equation becomes

$$\frac{d^2 y}{d\,\zeta^2} \;-\; 2\,\sin(1+\epsilon\,\zeta)\frac{d\,y}{d\,\zeta} \;+\; \epsilon\,\cos(1+\epsilon\,\zeta)\,y \;=\; 0. \tag{3.12}$$

The leading order solution for this will have the form $y \sim a + b\,\exp\{2\,\sin(1)\,\zeta\}$, where a and b are constants. For the boundary condition to be satisfied we need that $a + b = \epsilon^{-1/4}$. On the other hand, as ζ becomes large and negative, this solution should match with the behavior (as $x \to 1$) of the one in equation (3.11). Hence $a = 2\,\dfrac{\Gamma(1/4)}{\epsilon^{1/4}\sqrt{\pi}}\,\sqrt{\sin(1)}$. It follows that **the leading order behavior for the solution in the right boundary layer is given by**

$$y \sim \epsilon^{-1/4}\,\Big[\mu + (1-\mu)\,\exp\{2\,\sin(1)\,\zeta\}\Big],\quad \text{where}\;\; \mu = 2\,\frac{\Gamma(1/4)}{\sqrt{\pi}}\,\sqrt{\sin(1)}\;\;\text{and}\;\;\zeta = \frac{x-1}{\epsilon}. \tag{3.13}$$

REMARK: This problem was "cooked" so that the solution is "simple". If an $O(1)$ value for the boundary condition on $x = 1$ had been given, like $y(1) = 1$, then the expansion would have been a bit more complicated — why?

THE END.