Answers to Problem Set Number 5.

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1 Some asymptotic expansions (statement).

[**Part (a)**]. Let $y = y(x)$ have an asymptotic expansion of the form

$$y \sim \sum_{n=1}^{\infty} a_n\, x^n \sim a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \quad \text{for } 0 \leq x \ll 1, \text{ where } a_1 > 0. \tag{1.1}$$

Then we can write

$$x \sim \sum_{n=1}^{\infty} b_n\, y^n \sim b_1 y + b_2 y^2 + b_3 y^3 + \ldots, \quad \text{for } 0 \leq y \ll 1, \text{ where } b_1 > 0. \tag{1.2}$$

$\boxed{\textbf{FIND } b_1, b_2, \textbf{ and } b_3.}$

[**Part (b)**]. Let $y = y(x)$ have an asymptotic expansion of the form $y \sim \sum_{n=1}^{\infty} a_n\, x^{n/3}$, for $0 \leq x \ll 1$, where $a_1 > 0$. **Modify (1.2) appropriately, and calculate the first two terms in an expansion for x in terms of y, for $0 \leq y \ll 1$.**

[**Part (c)**]. Let $y = x - x^2 \ln x - \delta_1$ for $0 < x \ll 1$, where $\delta_1 = O(x^2)$. **Solve for x as a function of y, for $0 < y \ll 1$, with as many terms as possible — including an estimate for the error term in your solution.**

[Part (d)]. Find the first three terms in an expansion (valid for $0 < x \ll 1$) for the solutions of

$$\cos y + x - 1 + x^2 = 0, \quad \text{where } y \text{ is small.} \tag{1.3}$$

Solution to the some asymptotic expansions problem.

Solution to part a. Substitute equation (1.2) into the right hand side of equation (1.1), expand each of the powers of x, and collect equal powers of y. This yields:

$$\begin{aligned} y &= a_1 x + a_2 x^2 + a_3 x^3 + O(x^4) \\ &= a_1 \left(b_1 y + b_2 y^2 + b_3 y^3 + O(y^4)\right) + a_2 \left(b_1 y + b_2 y^2 + O(y^3)\right)^2 + a_3 \left(b_1 y + O(y^2)\right)^3 + O(y^4) \\ &= a_1 b_1 y + \left(a_1 b_2 + a_2 b_1^2\right) y^2 + \left(a_1 b_3 + 2 a_2 b_1 b_2 + a_3 b_1^3\right) y^3 + O(y^4), \end{aligned} \tag{1.4}$$

where we have used the (obvious) fact that $x = O(y)$ to exchange the $O(x^4)$ term in the first line by a $O(y^4)$ in the second line. Equating to zero the coefficients of each power of y then gives:

$$\boxed{b_1 = a_1^{-1}, \quad b_2 = -a_2\, a_1^{-3}, \quad \text{and} \quad b_3 = 2\, a_2^2\, a_1^{-5} - a_3\, a_1^{-4}.} \tag{1.5}$$

Solution to part b. It is clear that, in this case, the leading order behavior is given by .... $\boxed{x \sim (y/a_1)^3.}$

Hence, write $\boxed{x = (y/a_1)^3\,(1+z)}$ — where z is small, and substitute into the given asymptotic expansion for y. This yields .... $y \sim \sum_{n=1}^{\infty} a_n\, a_1^{-n}\, y^n\, (1+z)^{n/3}$.

Hence z **has an asymptotic expansion in integer powers of** y

$$z \sim \sum_{n=1}^{\infty} c_n\, y^n \quad \Longrightarrow \quad x \sim \left(\frac{y}{a_1}\right)^3 \sum_{n=0}^{\infty} c_n\, y^n, \quad \text{where } c_0 = 1. \tag{1.6}$$

Substituting this into the expansion for y yields:

$$\begin{aligned} y &= y\left(1 + c_1 y + O(y^2)\right)^{1/3} + a_2\, a_1^{-2}\, y^2\, \left(1 + O(y)\right)^{2/3} + O(y^3) \\ &= y + \left(\frac{1}{3} c_1 + a_2\, a_1^{-2}\right) y^2 + O(y^3) \quad \Longrightarrow \quad c_1 = -3\, a_2\, a_1^{-2}. \end{aligned} \tag{1.7}$$

Thus

$$\boxed{x = \left(\frac{y}{a_1}\right)^3 \left(1 - 3\, a_2\, a_1^{-2}\, y + O(y^2)\right).} \tag{1.8}$$

$\boxed{\textbf{Solution to part c.}}$ Since $y = x - x^2 \ln x - \delta_1$ — with $0 < x \ll 1$ and $\delta_1 = O(x^2)$, it is clear that $x \sim y$. Hence $\delta_1 = O(y^2)$. We now write $x = y + z$, where $|z| \ll y$, and substitute into the equation $0 = x - y - x^2 \ln x - \delta_1$, to get

$$0 = z - (y+z)^2 \ln(y+z) + O(y^2) = z - y^2 \ln y + O(y^2, z\,y\,\ln y) \tag{1.9}$$

where the the order term follows from the fact that the derivative of $y^2 \ln y$ is $O(y \ln y)$ for $0 < y \ll 1$. Thus $z \sim y^2 \ln y$, and we can rewrite (1.9) in the form

$$0 = z - (y+z)^2 \ln(y+z) + O(y^2) = z - y^2 \ln y + O(y^2) \tag{1.10}$$

as follows from the fact that $y^3 \ln^2 y \ll y^2$ for $0 < y \ll 1$. Therefore **the answer to part c is**

$$\boxed{x = y + y^2 \ln y + O(y^2).} \tag{1.11}$$

$\boxed{\textbf{Solution to part d.}}$ Since $\cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 - \frac{1}{720}y^6 + O(y^8)$, the equation can be written in the form

$$x + x^2 = \frac{1}{2}y^2 - \frac{1}{24}y^4 + \frac{1}{720}y^6 + O(y^8), \tag{1.12}$$

where y is small and $0 < x \ll 1$. Clearly $y \sim \pm\sqrt{2x}$. Thus we write $\boxed{y = \pm\sqrt{2x}\,(1+z)}$ — where z is small. This leads to

$$\begin{aligned}
x + x^2 &= x\,(1+z)^2 - \frac{1}{6}x^2\,(1+z)^4 + \frac{1}{90}x^3\,(1+z)^6 + O(x^4), \quad \Longrightarrow \\
\frac{7}{6}x^2 - \frac{1}{90}x^3 &= 2\,x\,z + x\,z^2 - \frac{2}{3}x^2\,z + O(x^4,\ x^3\,z,\ x^2\,z^2), \quad \Longrightarrow \\
\frac{7}{6}x - \frac{1}{90}x^2 &= 2\,z + z^2 - \frac{2}{3}x\,z + O(x^3),
\end{aligned}$$

where (in the last line) we divided by x, and used the (obvious) fact that $z = O(x)$ to simplify the order term. It should now be clear that z **has an expansion in integer powers of x, namely:** $\boxed{z = a\,x + b\,x^2 + \dots}$ Substituting this into the last equation yields:

$$\frac{7}{6}x - \frac{1}{90}x^2 = 2\,a\,x + \left(2\,b + a^2 - \frac{2}{3}a\right)x^2 + O(x^3).$$

Thus $a = 7/12$ and $b = 3/160$, so that

$$\boxed{y = \pm\sqrt{2\,x}\left(1 + \frac{7}{12}x + \frac{3}{160}x^2 + O(x^3)\right).} \tag{1.13}$$

2 Cole-Hopf transformation (statement).

Consider the equation for $u = u(x,t)$ (known as **Burgers'** equation)

$$u_t + 2uu_x = u_{xx}. \tag{2.1}$$

Introduce now $v = v(x,t)$ by $u = v_x$. **Show that one can choose v so that:**

$$v_t + v_x^2 = v_{xx}. \tag{2.2}$$

Let now $\Phi = e^{-v}$ (so that $v = -\ln\Phi$). **Show that Φ satisfies the heat equation:**

$$\Phi_t = \Phi_{xx}. \tag{2.3}$$

The net effect of all this is that the nonlinear equation (2.1) is linearized by the transformation $u = -\Phi_x/\Phi$. This transformation is known as the **Cole-Hopf transformation.**

Solution to the Cole-Hopf transformation problem.

Upon introduction of v via $u = v_x$, the equation becomes

$$\left(v_t + v_x^2\right)_x = v_{xxx} \quad \Longrightarrow \quad v_t + v_x^2 = v_{xx} + C(t), \tag{2.4}$$

where $C = C(t)$ is some function of time. However, v is defined up to an arbitrary function of time by the equation $u = v_x$. Hence we can change $v \to v + \int^t C(s)\,ds$, to set $C \equiv 0$ in (2.4), thus obtaining equation (2.2). **This finishes the first part of the problem.**

Let now $\Phi = e^{-v}$. Substituting $v = -\ln\Phi$ into equation (2.2) yields

$$-\frac{\Phi_t}{\Phi} + \left(-\frac{\Phi_x}{\Phi}\right)^2 = -\frac{\Phi_{xx}}{\Phi} + \frac{\Phi_x^2}{\Phi^2} \quad \Longrightarrow \quad \Phi_t = \Phi_{xx}. \tag{2.5}$$

This finishes the second part of the problem.

3 Pendulum Small Amplitude Expansion (statement).

The (dimension-less) equation for a pendulum is

$$\frac{d^2y}{dt^2} + \sin y = 0. \tag{3.1}$$

Consider now a small amplitude expansion $y \sim \sum_{n=1}^{\infty} \epsilon^n y_n(t)$, for the solution with initial conditions $y(0) = 0$ and $\frac{dy}{dt}(0) = \epsilon$ — where $0 < \epsilon \ll 1$. Calculate a few terms in the expansion, and answer the following questions:

(a) Is the expansion valid for all $0 < t < \infty$? If no, how large can t be?

(b) If the answer to part (a) is no, can you explain the reason why a regular expansion does not work for this problem?

Solution to Pendulum Small Amplitude Expansion.

Since $\sin y = y - \frac{1}{3!} y^3 + \frac{1}{5!} y^5 + \dots$, substitution of $y \sim \sum_{n=1}^{\infty} \epsilon^n y_n(t)$ into equation (3.1) yields

$\boxed{\textbf{At } O(\epsilon).}$ .... $\ddot{y}_1 + y_1 = 0$, with $y_1(0) = 0$ and $\dot{y}_1(0) = 1$.

Hence .... $\boxed{y_1 = \sin t.}$

$\boxed{\textbf{At } O(\epsilon^2).}$ .... $\ddot{y}_2 + y_2 = 0$, with $y_2(0) = \dot{y}_2(0) = 0$.

Hence .... $\boxed{y_2 = 0.}$

$\boxed{\textbf{At } O(\epsilon^3).}$ .... $\ddot{y}_3 + y_3 = \frac{1}{6} y_1^3 = \frac{1}{8} \sin t - \frac{1}{24} \sin 3t$, with $y_3(0) = \dot{y}_3(0) = 0$.

Hence .... $\boxed{y_3 = \frac{1}{192} \sin 3t - \frac{1}{16} t \cos t + \frac{3}{64} \sin t.}$

Important observations:

IO-1: $y_n = 0$ **for all** n **even.**

IO-2: For $n = 4m - 1$, the largest term in y_n (as $t \to \infty$) is proportional to $t^{2m-1} \cos t$.

IO-3: For $n = 4m + 1$, the largest term in y_n (as $t \to \infty$) is proportional to $t^{2m} \sin t$.

Proof of IO-1. *Let $y = y(\epsilon, t)$ be the exact solution to the problem. Then, though we are only looking at the case $\epsilon > 0$, it should be clear that $y(-\epsilon, t) = -y(-\epsilon, t)$. Hence only odd powers of ϵ should appear in the expansion.*

Proof (sketch) of IO-2 and IO-3. *Let $n = 2m + 1 \geq 5$. Then it is easy to see that the n-th term in the expansion is defined by an equation of the form*

$$\ddot{y}_n + y_n = R_n(y_1, y_3 \dots y_{n-2}), \tag{3.2}$$

where R_n is a sum of products of the form $y_{n_1}\, y_{n_2} \dots y_{n_q}$ where $n_1 + n_2 + \ldots + n_q = n$, and the n_j's are odd — with $1 \le n_j < n$. An easy inductive argument then shows that

$$\left.\begin{array}{l} y_n \text{ is a sum of terms of the form } \left(P_s(t)\, e^{i\,(2s+1)\,t} + c.c.\right), \text{ with } 0 \le s \le m, \text{ where} \\ c.c. \text{ denotes the complex conjugate, and } P_s(t) \text{ is a polynomial of degree } \le m \\ \text{— actually, degree}(P_s) = m - s, \text{ but this is a little harder to prove.} \end{array}\right\} \tag{3.3}$$

This follows because the general solution of $\ddot{y} + y = P(t)\, e^{i\mu t}$, where P is a polynomial, has the form $y = Q(t)\, e^{i\mu t} +$ homogeneous solution, , where Q is a polynomial and **(i)** *degree(Q) = degree(P) if $\mu \neq \pm 1$.* **(ii)** *degree(Q) = degree(P) + 1 if $\mu = \pm 1$.*

In particular, the term $\dfrac{1}{2}\, y_{n-2}\, y_1^2$ on the right hand side in equation (3.3) — which arises because of the term $-\dfrac{1}{3!}\, y^3$ in the expansion for $\sin y$ — shows that

$$\left.\begin{array}{l} y_n \text{ has the term } \left(a_n\, t^m\, e^{i\,t} + c.c.\right), \text{ where } a_n \text{ is a constant. In fact } a_1 = \dfrac{1}{2i}, \\ a_3 = -\dfrac{1}{32}, \text{ and } a_n = \dfrac{1}{2im}\, |a_1^2|\, a_{n-2} \text{ for } n > 3. \text{ This because } \dfrac{1}{2}\, y_{n-2}\, y_1^2 \text{ gen-} \\ \text{erates the forcing term } a_{n-2}\, |a_1^2|\, t^{m-1}\, e^{it} \text{ on the right hand side in (3.2).} \end{array}\right\} \tag{3.4}$$

IO-2 and IO-3 follow, trivially, from (3.4).

Solution to part a.

IO-2 and **IO-3** show that y_n, with $n = 2m + 1$, is $O(t^m)$ for $t \gg 1 \implies \epsilon^n\, y_n = O(\epsilon\, (\epsilon^2\, t)^m)$. Hence:

$$\boxed{\textbf{The expansion is NOT valid for all } 0 < t < \infty. \textbf{ It is valid only as long as } 0 < t \ll e^{-2}.}$$

Solution to part b.

The solution to this problem can be written in terms of quadratures, as follows: Multiply equation (3.1) by $2\,\dot{y}$ and integrate. This yields:

$$\left(\frac{dy}{dt}\right)^2 = 2\,(\cos y - 1) + \epsilon^2, \quad \text{with } y(0) = 0, \tag{3.5}$$

where the initial conditions have been used to pick the integration constant. The solution to this equation is a periodic function, oscillating between the two smallest zeros of $\cos y = 1 - \dfrac{1}{2}\,\epsilon^2$, i.e. $y = \pm y_m(\epsilon)$, where $y_m = \epsilon + O(\epsilon^2)$. The period T, is given by

$$T = 2\int_{-y_m}^{+y_m} \frac{dy}{\sqrt{2\,(\cos y - 1) + \epsilon^2}} \approx 2\int_{-\epsilon}^{+\epsilon} \frac{dy}{\sqrt{\epsilon^2 - y^2}} = 2\int_{-1}^{+1} \frac{ds}{\sqrt{1 - s^2}} = 2\,\pi. \tag{3.6}$$

The solution is even relative to $t = \pm\dfrac{T}{4}$, and it is given (implicitly) by

$$t = \int_0^{y(t)} \frac{ds}{\sqrt{2\,(\cos s - 1) + \epsilon^2}} \quad \text{for} \quad |t| \leq \frac{T}{2}. \tag{3.7}$$

The important point, though, is that **the period is a function of ϵ (i.e. $T = T(\epsilon)$), and only approximately equal to $2\,\pi$. An approximation to the solution valid for large times MUST incorporate this fact. Such an approximation is**[1]

$$y = \epsilon \sin(\omega\, t) + \frac{1}{192}\,\epsilon^3 \sin(3\,\omega\, t) + O(\epsilon^5), \tag{3.8}$$

where $\omega = \dfrac{2\,\pi}{T} = 1 + \omega_1\,\epsilon^2 + \omega_2\,\epsilon^4 + \ldots$ and T is given by (3.6). As long as $\epsilon^2\,t$ is small, we can expand the sines in equation (3.8) to obtain:

$$y = \epsilon \sin t + \epsilon^3\,\omega_1\,t\,\cos t + + \frac{1}{192}\,\epsilon^3 \sin(3\,t) + O(\epsilon^5,\ \epsilon^5\,t^2,\ \epsilon^5\,t). \tag{3.9}$$

Since one can show that $\omega_1 = -\dfrac{1}{16}$ **(can you show this?)**, this agrees with the expansion obtained earlier in this problem. The terms in **IO-1** and **IO-2** follow from the higher order terms in the Taylor expansion of $\sin(\omega\,t)$ for $\omega\,t \approx t$.

THE END.

[1] This can be obtained using the exact solution in (3.5 – 3.7).