Answers to Problem Set Number 4.
18.305 — MIT (Fall 2005).

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1 Linear KdV equation asymptotics (statement).

The deviations u from a flat surface produced by a very small amplitude, uni-directional, long gravity wave disturbance in a water channel are described by the linear Korteweg de-Vries (KdV) equation. This equation can be written in the (dimension-less) form

$$u_t - \frac{1}{3}\,\epsilon^2\,u_{xxx} = 0, \quad \text{where} \quad 0 < \epsilon = h/\lambda \ll 1, \tag{1.1}$$

x is the length coordinate along the channel — in a frame moving at the infinite wavelength wave-speed $\sqrt{gh}$, λ is the wavelength, h is the channel depth, and g is the acceleration of gravity.

Given initial conditions for u, the solution to equation (1.1) above can be written in terms of Fourier Transforms, as follows:

$$u = \int_{-\infty}^{\infty} \exp\left\{ i\left(k\,x - \frac{\epsilon^2}{3}\,k^3\,t \right) \right\}\, g(k)\,dk, \tag{1.2}$$

where

$$g(k) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} u(x,0)\,\exp\left(-\,i\,k\,x\right)\,dx. \tag{1.3}$$

Consider now the situation where the initial data are localized in a very small region, of size ϵ. Namely, assume that

$$u(x,0) = U\left(\frac{x}{\epsilon}\right), \tag{1.4}$$

where $U = U(\zeta)$ decays very fast as $|\zeta| \to \infty$.

Part (a): Show that the Fourier Transform g — as given by the formula in (1.3) — for the initial data in (1.4), has the form $\boxed{g = \epsilon\, G\,(\epsilon\, k)}$ — for some function G.

Part (b): Use the result in part (a) to write the solution to the linear KdV equation in the form

$$u = \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\epsilon}\,\theta\right\} G\, d\kappa, \tag{1.5}$$

for appropriately selected variables, where $0 < \epsilon \ll 1$ shows up in the solution only in the exponent (as indicated in the formula above). **You need equation (1.5) to do parts (c-e) below.**

Part (c): Calculate the leading order behavior of the solution for fixed $x > 0$ and $t > 0$, as $\epsilon \downarrow 0$.

Part (d): Show that, for fixed $x < 0$ and $t > 0$, the solution vanishes to all orders in ϵ, as $\epsilon \downarrow 0$.

Part (e): Calculate the leading order behavior of the solution for fixed $t > 0$ and $x \approx 0$, as $\epsilon \downarrow 0$.

Hints for parts c-e: *Stationary phase, integration by parts, and Airy.*

Solution to the linear KdV equation asymptotics problem.

Part (a): From equations (1.3) and (1.4), we can write:

$$\begin{aligned} g(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U\left(\frac{x}{\epsilon}\right) \exp\left(-\,i\,k\,x\right)\, dx \\ &= \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty} U(X)\, \exp\left(-\,i\,\epsilon\,k\,X\right)\, dX \;=\; \epsilon\, G(\epsilon\, k), \end{aligned} \tag{1.6}$$

where $x = \epsilon\, X$ and $\boxed{G = G(\kappa)}$ **is the Fourier Transform of** $\boxed{U = U(X).}$

Part (b): From equations (1.2) and (1.6), we can write:

$$\begin{aligned} u &= \epsilon\int_{-\infty}^{\infty} \exp\left\{i\left(k\,x - \frac{\epsilon^2}{3}\,k^3\,t\right)\right\} G(\epsilon\, k)\, dk \\ &= \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\epsilon}\left(\kappa\, x - \frac{1}{3}\,\kappa^3\,t\right)\right\} G(\kappa)\, d\kappa \;=\; \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\epsilon}\,\theta\right\} G(\kappa)\, d\kappa, \end{aligned} \tag{1.7}$$

where $\kappa = \epsilon\, k$ and $\boxed{\theta = \kappa\, x - \frac{1}{3}\kappa^3\, t.}$

Part (c): From equation (1.7) we calculate the leading order behavior of u for $x > 0$ and $t > 0$, in the limit $\epsilon \downarrow 0$. For this we use the *Stationary Phase* method.

The stationary phase points κ_s are given by the solutions to the equation

$$0 = \frac{\partial\,\theta}{\partial\,\kappa} = x - \kappa^2\, t. \tag{1.8}$$

Hence $\boxed{\kappa_s = \pm\sqrt{x/t}.}$ Each of these two points will produce a dominant contribution to the solution. These two contributions are complex conjugates of each other when the solution is real valued — i.e. when $G(-\kappa) = G(\kappa)^*$, where the asterisk denotes the complex conjugate.

The contribution from the stationary point $\kappa_s = \pm\sqrt{x/t}$ is given by

$$\begin{aligned} u_\pm &= G(\kappa_s)\, \exp\left\{\frac{i}{\epsilon}\,\theta_s\right\} \int_{-\infty}^{-\infty} \exp\left\{\frac{i}{2\,\epsilon}\,\theta_s''\,(\kappa-\kappa_s)^2\right\} d\kappa \\ &= \sqrt{\frac{2\,\pi\,\epsilon}{|\theta_s''|}}\, G(\kappa_s)\, \exp\left\{\frac{i}{\epsilon}\,\theta_s + i\,\sigma\,\pi/4\right\} \\ &= \frac{\sqrt{\pi\,\epsilon}}{(x\,t)^{1/4}}\, G\left(\pm\sqrt{x/t}\right)\, \exp\left\{\mp i\,\pi/4 \pm i\,\epsilon^{-1}\,(2/3)\,x^{3/2}\,t^{-1/2}\right\}. \end{aligned} \tag{1.9}$$

where $\boxed{\theta_s = \pm(2/3)\,x^{3/2}\,t^{-1/2}}$ is the value of θ at $\kappa = \kappa_s$, $\boxed{\theta_s'' = \mp 2\,\sqrt{x\,t}}$ is the value of $\theta_{\kappa\kappa}$ at $\kappa = \kappa_s$, and $\boxed{\sigma = \mp 1}$ is the sign of θ_s''. **Note that:** to perform the integral in the first line of the equation above, the contour of integration is changed from the real axis to the path $\kappa = \kappa_s + e^{i\,\sigma\,\pi/4}\,t$ — where $-\infty < t < \infty$.

The leading order behavior of the solution is then given by .... $\boxed{u \sim u_+ + u_-,}$ where $u_\pm$ are given by (1.9) above.

Part (d): We use equation (1.7) to show that, for $x < 0$ and $t > 0$, in the limit $\epsilon \downarrow 0$, the solution u vanishes to all orders in ϵ. We do this using *Integration by parts.*

When $x < 0$ and $t > 0$, θ is a monotone decreasing function of κ, as follows from (1.8). Hence:

$$u \;=\; \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\epsilon}\,\theta\right\} G\, d\kappa,$$

$$= \; -i\,\epsilon \int_{-\infty}^{\infty} G\,\lambda\,d\left(\exp\left\{\frac{i}{\epsilon}\,\theta\right\}\right)$$

$$= \; +i\,\epsilon \int_{-\infty}^{\infty} \frac{\partial(G\,\lambda)}{\partial\kappa}\,\exp\left\{\frac{i}{\epsilon}\,\theta\right\}\,d\kappa, \tag{1.10}$$

where $\lambda = \left(\dfrac{\partial\theta}{\partial\kappa}\right)^{-1}$ is well defined. This shows that $u = O(\epsilon)$. However, as long as G has derivatives of all orders in κ, this process can be repeated, to show that $\boxed{u = O(\epsilon^n)}$ — for any $n > 0$.

Part (e): When $x \approx 0$ and $t > 0$ fixed, the zeros of equation (1.8) are both near $\kappa = 0$. Hence the dominant contribution to the solution u arises from $\kappa \approx 0$, and (1.7) yields

$$u \sim G(0) \int_{-\infty}^{\infty} \exp\left\{\frac{i}{\epsilon}\left(\kappa\,x - \frac{1}{3}\,\kappa^3\,t\right)\right\}\,d\kappa.$$

Changing variables $\kappa = -(\epsilon/t)^{1/3}\,z$, we obtain

$$u \sim G(0)\,(\epsilon/t)^{1/3} \int_{-\infty}^{\infty} \exp\left\{i\left(z\,\zeta + \frac{1}{3}\,z^3\,t\right)\right\}\,dz,$$

where $\boxed{\zeta = -\dfrac{x}{\epsilon^{2/3}\,t^{1/3}}.}$ It follows that

$$\boxed{u \sim 2\,\pi\,G(0)\,(\epsilon/t)^{1/3}\,\mathrm{Ai}(\zeta).} \tag{1.11}$$

Note that:

(A) For $\zeta \gg 1$ (1.11) yields u exponentially small, which agrees with the result in part **d**.

(B) For $-\zeta \gg 1$ (1.11) yields, upon use of the Airy function known asymptotic behavior:

$$u \sim \frac{2\,\sqrt{\pi\,\epsilon}}{(x\,t)^{1/4}}\,G(0)\,\sin\left\{\epsilon^{-1}\,(2/3)\,x^{3/2}\,t^{-1/2} + \pi/4\right\}.$$

This matches the behavior obtained from part **c** for x small.

2 An eigenvalue problem (statement).

Let $0 < \epsilon \ll 1$, and consider the eigenvalue problem

$$-\epsilon^2\,y'' + V(x)y = Ey, \quad -\infty < x < \infty, \tag{2.1}$$

where V is given by: $V(x) = 1$ for $|x| > 1$ and $V(x) = 0$ for $|x| < 1$. The solutions must vanish as $x \to \pm\infty$ and must be continuous with continuous first derivatives.

Calculate the eigenfunctions and eigenvalues for this problem exactly. Compare the eigenvalues thus obtained with the answer provided by equation (10.5.6) in Bender & Orszag's book. EXPLAIN any discrepancies.

HINT: (i) Eigenvalues exist only in the range $0 \leq E \leq 1$ **(why? – there is an easy argument!)** (ii) For $0 < E < 1$ the transformation $1 - 2E = \cos\theta_0$ and $2\sqrt{(1-E)\,E} = \sin\theta_0$, with $0 < \theta_0 < \pi$, is useful to write the equation for the eigenvalues. The cases $E = 0$ and $E = 1$ must be treated separately.

Solution to the eigenvalue problem.

From the uniqueness of the solutions to ode initial value problems, it follows that:

$$\boxed{\begin{array}{l}\textbf{Let } \Phi = \Phi(x) \textbf{ be a solution to equation (2.1). Suppose that there is a point } x_m \\ \textbf{such that } \Phi(x_m) = \Phi'(x_m) = 0\textbf{. Then the solution vanishes identically: } \Phi \equiv 0.\end{array}} \tag{2.2}$$

$\boxed{\textbf{(2.a) Case } E > 1.}$ Let Φ be an eigenfunction in this case (if any exist). Then:

$\epsilon^2\,\Phi'' = -(E-1)\,\Phi$ for $|x| > 1 \implies \Phi$ is oscillatory (sine and/or cosine) for $|x| > 1$. Hence, for Φ to vanish as $|x| \to \infty$, it must be $\Phi \equiv 0$ for $|x| > 1$. Then, from (2.2), Φ vanishes everywhere. Since eigenfunctions cannot vanish identically ..... $\boxed{\textbf{There are no eigenvalues such that } E > 1.}$

$\boxed{\textbf{(2.b) Case } E = 1.}$ Let Φ be an eigenfunction in this case (if any exist). Then:

For $|x| > 1$, $\Phi \equiv 0$ — since $\Phi'' = 0$ for $|x| > 1$, and Φ must vanish for $|x| \to \infty$. The same argument as in **(2.a)** then shows that .... $\boxed{E = 1 \textbf{ is not an eigenvalue.}}$

$\boxed{\textbf{(2.c) Case } 0 < E < 1.}$ Let Φ be an eigenfunction in this case (if any exist). Then:

(2.c.1) For $x < -1$, $\Phi = A\,\exp\left\{\dfrac{\sqrt{1-E}}{\epsilon}\,(1+x)\right\}$ — for some constant A.

Without loss of generality, we can assume that $\boxed{A = 1.}$

(2.c.2) For $x > +1$, $\Phi = B\,\exp\left\{\dfrac{\sqrt{1-E}}{\epsilon}\,(1-x)\right\}$ — for some constant B.

(2.c.3) For $|x| < 1$, $\Phi = a \sin\left\{\dfrac{\sqrt{E}}{\epsilon}(1+x)\right\} + b \cos\left\{\dfrac{\sqrt{E}}{\epsilon}(1+x)\right\}$ — for some constants a, b.

(2.c.4) Continuity of Φ and its derivative at $x = -1$ requires $b = 1$ and $a = \dfrac{\sqrt{1-E}}{\sqrt{E}}$.

(2.c.5) Continuity of Φ and its derivative at $x = +1$ requires $B = a \sin\theta + \cos\theta$ and

$$(1 - 2\,E)\,\sin\theta + 2\,\sqrt{1-E}\,\sqrt{E}\,\cos\theta = 0, \quad \text{where} \quad \theta = 2\,\sqrt{E}/\epsilon. \tag{2.3}$$

$\boxed{\textbf{This is the equation determining the eigenvalues.}}$

(2.c.6) Define $\boxed{0 < \theta_0 < \pi}$ by $1 - 2\,E = \cos\theta_0$ and $2\,\sqrt{1-E}\,\sqrt{E} = \sin\theta_0$.
Then equation (2.3) becomes $\sin(\theta + \theta_0) = 0$, with solution $\theta + \theta_0 = n\,\pi$ — where n is an integer such that $\boxed{0 < n < 1 + \dfrac{2}{\epsilon\,\pi}}$ (since $0 < \theta < 2/\epsilon$ and $0 < \theta_0 < \pi$). It follows that

$$\frac{2\,\sqrt{E}}{\epsilon} = n\,\pi - \theta_0 = \left(n - \frac{1}{2}\right)\pi + \arctan\left(\frac{1 - 2\,E}{2\,\sqrt{1-E}\,\sqrt{E}}\right) \tag{2.4}$$

is an $\boxed{\textbf{alternative form of the equation for the eigenvalues.}}$

Equation (2.4) cannot be solved explicitly. On the other hand $g(E) = \epsilon^{-1}\,2\,\sqrt{E} + \theta_0$ is a monotone increasing function of E, with $g(0) = 0$ and $g(1) = \pi + 2\,\epsilon^{-1}$. Hence, for every integer n in the range $0 < n < 1 + \dfrac{2}{\epsilon\,\pi}$, equation (2.4) has exactly one solution.

$\boxed{\textbf{(2.d) Case } E = 0.}$ Let Φ be an eigenfunction in this case (if any exist). Then

(2.d.1) For $x < -1$, $\Phi = A \exp\left\{\dfrac{1+x}{\epsilon}\right\}$ — for some constant A.

(2.d.2) For $x > +1$, $\Phi = B \exp\left\{\dfrac{1-x}{\epsilon}\right\}$ — for some constant B.

(2.d.3) For $|x| < 1$, $\Phi = a + b\,x$ — for some constants a and b.

(2.d.4) Continuity of Φ and its derivative at $x = -1$ requires $a = (1 + \epsilon^{-1})\,A$ and $b = +A/\epsilon$.

(2.d.5) Continuity of Φ and its derivative at $x = +1$ requires $a = (1 + \epsilon^{-1})\,B$ and $b = -B/\epsilon$.

Items **(2.d.4)** and **(2.d.5)** are compatible with $A = B = a = b = 0$ only. Since eigenfunctions cannot vanish identically, we conclude that .... $\boxed{E = 0 \textbf{ is not an eigenvalue.}}$

(2.e) Case $E < 0$. Let Φ be an eigenfunction in this case (if any exist). Then:

Since Φ vanishes as $|x| \to \infty$ (and Φ cannot vanish identically), Φ must have either a maximum or a minimum. Without loss of generality, we can assume that Φ has a maximum at some point $-\infty < x_m < \infty$, where $\Phi(x_m) > 0$. But then $\Phi''(x_m) = (V(x_m) - E)\,\epsilon^{-2}\,\Phi(x_m) > 0$, a contradiction since Φ has a maximum at x_m. Thus .... **There are no eigenvalues such that $E < 0$.**

(2.f) Comparison with WKB: Bender & Orszag's formula (10.5.6).

Application of this formula leads to the eigenvalue equation

$$\frac{2\sqrt{E}}{\epsilon} = \left(n + \frac{1}{2}\right)\pi + O(\epsilon) \tag{2.5}$$

for $0 < E < 1$. Comparison with the exact formula (2.4) shows that this last equation would be correct only if $\theta_0 + \pi/2 = O(\epsilon)$ — which is clearly incorrect. **Hence there is a discrepancy between the exact formula for the eigenvalues, and the prediction of equation (10.5.6) in Bender and Orszag's book.**

The explanation for the discrepancy is very simple. Equation (10.5.6) is derived using the turning point connection formulas valid for SIMPLE zeros in $(V - E)$. Here, however, we have discontinuous jumps where $(V - E)$ switches from positive to negative. Hence the discrepancy.

3 Connection formulas (statement).

Let $0 < \epsilon \ll 1$, and consider the ODE problem

$$\epsilon^2\, y'' = Q(x) y, \tag{3.1}$$

where $Q(x) > 0$ for $x > 0$, and $Q(x) < 0$ for $x < 0$. However, Q does not have a simple zero at $x = 0$. Instead, assume that Q has a jump discontinuity at $x = 0$ — with Q smooth on each side, with left and right limits for it and all its derivatives. **Calculate the connection formulas across the origin, for the WKB approximations for the problem stated above.**

Solution to the connection formulas problem.

$$\boxed{\textbf{For simplicity, we will assume that } \lim_{x\to 0+} Q(x) = Q_+ > 0 \textbf{ and } \lim_{x\to 0-} Q(x) = Q_- < 0.}$$

The case where either of these limits vanishes is much more complicated, for then the solution to the problem requires asymptotic information as to how the limit is achieved. For example, if $Q = O(x^\mu)$ for $x > 0$, then the answer depends on the value of μ.

For $x > 0$, the WKB approximation to the (exponentially decaying) solution has the form

$$y = c\, Q^{-1/4}\ (1 + O(\epsilon))\ \exp\left\{-\frac{1}{\epsilon}\int_0^x \sqrt{Q(s)}\, ds\right\}, \tag{3.2}$$

where c is some constant. Because $Q_+ > 0$, this approximation is valid down to $x = 0+$.

For $x < 0$, the WKB approximation to the solution has the form

$$y = a\,|Q|^{-1/4}\ (1 + O(\epsilon))\ \sin\left\{\frac{1}{\epsilon}\int_x^0 \sqrt{|Q(s)|}\, ds\right\} + b\,|Q|^{-1/4}\ (1 + O(\epsilon))\ \cos\left\{\frac{1}{\epsilon}\int_x^0 \sqrt{|Q(s)|}\, ds\right\}, \tag{3.3}$$

where a and b are some constants. Because $Q_- < 0$, this approximation is valid up to $x = 0-$.

Continuity of y at $x = 0$ requires .... $\boxed{c\ (1 + O(\epsilon))\ Q_+^{-1/4} = b\ (1 + O(\epsilon))\ |Q_-|^{-1/4}.}$

Continuity of y' at $x = 0$ requires .... $\boxed{c\ (1 + O(\epsilon))\ Q_+^{+1/4} = a\ (1 + O(\epsilon))\ |Q_-|^{+1/4}.}$

Hence the **WKB connection formulas across a discontinuity** are

$$a = |Q_+/Q_-|^{1/4}\ (1 + O(\epsilon))\ c \quad \text{and} \quad b = |Q_-/Q_+|^{1/4}\ (1 + O(\epsilon))\ c. \tag{3.4}$$

4 Klein Gordon Reflection & Transm. Coeff. (statement).

Consider a string under tension, attached to a (very stiff) elastic bed, with the stiffness of the elastic bed varying along the string length. The small deviations u from equilibrium (straight string) then satisfy the Klein Gordon equation, which can be written in the dimension-less form

$$u_{tt} - u_{xx} + \frac{1}{\epsilon^2}V(x)u = 0, \quad \text{where} \quad 0 < \epsilon \ll 1. \tag{4.1}$$

Assume now that V is a smooth even function, vanishes outside some finite interval $|x| < a$, and is positive inside it. In fact, assume that $dV/dx > 0$ for $-a < x < 0$, with a single maximum at the origin, where $V(0) = 1$. In this case the solutions outside of the interval $|x| < a$ must have the simple form: $\boxed{u = f(x-t) + g(x+t)}$ — for some functions f and g. In other words, they are the superposition of two steady waves with unit speed: one moving to the right and the other moving to the left.

In particular, we now look at the situation where a single frequency wave is launched from $x = -\infty$, and we want to compute how much is reflected and how much is transmitted. This problem can be formulated mathematically as follows: Find the solution defined by the following properties

(A) For $x < -a$ .... $\boxed{u = \exp\left\{i\,\frac{\omega}{\epsilon}\,(t-x)\right\} + R\,\exp\left\{i\,\frac{\omega}{\epsilon}\,(t+x)\right\},}$

(B) For $x > +a$ .... $\boxed{u = T\,\exp\left\{i\,\frac{\omega}{\epsilon}\,(t-x)\right\},}$

where ω, R, and T are constants, with ω given (and real) and the other two are to be determined.

(a) Use WKB techniques to produce approximate expressions for R and T.

(b) When $V = 1 - |x|$ for $|x| < 1$, give explicit (approximate) formulas for R and T.

EXTRA CREDIT (15 points): In case (b) above the problem can be solved exactly, and the WKB approximation can be checked. Do this. **Important:** *The extra credit applies ONLY if you do this right. You can, and will, get NEGATIVE credit if you botch this up, even minimally. Hence: hand this part in ONLY if you are 100% certain of your answer.*

Solution to the Klein Gordon R. & T. Coefficient problem.

Separate time in the problem, and write $u = e^{i\,\omega\,t/\epsilon}\,\Phi(x)$. Then Φ must satisfy

$$\epsilon^2\,\Phi'' = (V(x) - \omega^2)\,\Phi, \tag{4.2}$$

$$\Phi = \exp\left\{-i\,\frac{\omega}{\epsilon}\,x\right\} + R\,\exp\left\{+i\,\frac{\omega}{\epsilon}\,x\right\}, \quad \text{for } x < -a, \tag{4.3}$$

$$\Phi = T\,\exp\left\{-i\,\frac{\omega}{\epsilon}\,x\right\}, \quad \text{for } x > +a, \tag{4.4}$$

where the objective is to find $R = R(\omega)$ and $T = T(\omega)$, for $-\infty < \omega \neq 0 < \infty$. **Note that the problem is not defined for $\omega = 0$,** since then the solutions are not oscillatory for $|x| > a$ (i.e. there are no zero frequency "waves"). Hence we **must exclude from consideration the case $\omega = 0$,** as noted above.

We observe that

(4.a) **We need only consider** $0 < \omega < \infty$**, since** $\boxed{R(-\omega) = \overline{R(\omega)}}$ **and** $\boxed{T(-\omega) = \overline{T(\omega)}}$ — where the over-bar indicates the complex conjugate. This follows easily upon taking the complex conjugate of equations (4.2 – 4.4).

(4.b) **For** $0 < \omega < \infty$**, with** $E = \omega^2$ **and** $\sqrt{E} = \omega$, the **problem above in equations (4.2 – 4.4)** becomes **exactly the tunneling problem** considered in the lectures and in Bender and Orszag's book. Namely:

(4.c) **Tunneling problem:** Let $V = V(x)$ be a *smooth single bumped positive potential, vanishing at infinity and with maximum value* $V_m > 0$. Then solve, for $0 < E < \infty$,

$$\begin{aligned}
\epsilon^2\,\Phi'' &= (V(x) - E)\,\Phi, \\
\Phi &\sim \exp\left\{-i\,\frac{\sqrt{E}}{\epsilon}\,x\right\} + R\,\exp\left\{+i\,\frac{\sqrt{E}}{\epsilon}\,x\right\}, \quad \text{as } x \to -\infty, \\
\Phi &\sim T\,\exp\left\{-i\,\frac{\sqrt{E}}{\epsilon}\,x\right\}, \qquad\qquad \text{as } x \to +\infty.
\end{aligned}$$

(4.d) **Tunneling problem** $0 < \epsilon \ll 1$ **WKB approximation to the solution:** *For* $0 < \epsilon < \infty$*, WKB provides an approximation for the solution, as follows:*

(4.d.1) For $E > V_m$, $T \sim 1$ and R **vanishes,** *to all orders in the WKB approximation.*

(4.d.2) For $0 < E < V_m$, let $-\infty < A = A(E) < B = B(E) < \infty$ *be the two solutions to the equation* $V = E$. Then define $\mu_L = \mu_L(E)$, $\mu_M = \mu_M(E)$, and $\mu_R = \mu_R(E)$ by

$$\begin{aligned}
\mu_L &= \int_{-\infty}^{A} \left(\sqrt{E - V(x)} - \sqrt{E}\right) dx + A\,\sqrt{E}, \\
\mu_M &= \int_{A}^{B} \sqrt{V(x) - E}\,dx, \qquad \text{and} \\
\mu_R &= \int_{B}^{+\infty} \left(\sqrt{E - V(x)} - \sqrt{E}\right) dx - B\,\sqrt{E}.
\end{aligned}$$

Then $\boxed{R \sim \exp\left\{i\left(\frac{\pi}{2} - 2\,\frac{\mu_L}{\epsilon}\right)\right\}}$ and $\boxed{T \sim \exp\left\{-\frac{1}{\epsilon}\,\left[\mu_M + i\,\left(\mu_L + \mu_R\right)\right]\right\}.}$

(4.d.3) Notice that these approximations breakdown for $E \approx 0$ **and** $E \approx V_m$**.** This because the *problem becomes singular for* $E = 0$, and because the two *turning points* A *and* B *merge for* $E = V_m$.

Solution to part a. Using the results in items **(4.a – 4.d)** above, we can now write an **approximation to the solution of equations (4.2 – 4.4), valid for** $\boxed{0 < \epsilon \ll 1.}$ Namely:

(4.e) For $1 < \omega < \infty$, $T \sim 1$ and R **vanishes,** *to all orders in the WKB approximation.*

(4.f) For $0 < \omega < 1$, let $B = B(\omega)$ be the *unique solution to the equation* $V(x) = \omega^2$ *in the interval* $0 < x < a$. Then

$$\boxed{R \sim \exp\left\{i\left(\frac{\pi}{2} - 2\,\frac{\mu}{\epsilon}\right)\right\}} \quad \textbf{and} \quad \boxed{T \sim \exp\left\{-\frac{1}{\epsilon}\left(\sigma + 2\,i\,\mu\right)\right\},} \tag{4.5}$$

where

$$\boxed{\mu = \int_B^a \sqrt{\omega^2 - V(x)}\,dx - a\,\omega} \quad \textbf{and} \quad \boxed{\sigma = 2\int_0^B \sqrt{V(x) - \omega^2}\,dx.} \tag{4.6}$$

In these equations we have used that V is even — e.g.: $A = -B$ and $\mu_L = \mu_R = \mu$.

(4.g) $R(-\omega) = \overline{R(\omega)}$ and $T(-\omega) = \overline{T(\omega)}$.

(4.h) **These approximations break down for $\omega \approx 0$ and $\omega \approx 1$.**

Solution to part b. **Assume now that $a = 1$, with $V = 1 - |x|$ for $|x| < 1$.** Then, in equations (4.5 – 4.6) above, we have: $\boxed{B = 1 - \omega^2,\ \mu = \frac{2}{3}\,\omega^3 - \omega, \text{ and } \sigma = \frac{4}{3}\left(1 - \omega^2\right)^{3/2}.}$

Thus

$$R \sim i\,\exp\left\{-\frac{2\,i}{\epsilon}\left(\frac{2}{3}\,\omega^3 - \omega\right)\right\} \quad \text{and} \quad T \sim \exp\left\{-\frac{4}{3\epsilon}\,(1-\omega^2)^{3/2} - \frac{2\,i}{\epsilon}\left(\frac{2}{3}\,\omega^3 - \omega\right)\right\}, \tag{4.7}$$

for $0 < \omega < 1$. In particular, note that:

For $\omega = 0$, we get $\mu = 0$ and $\sigma = 4/3$.

For $\omega = 1$, we get $\mu = -1/3$ and $\sigma = 0$.

Though the approximation is not to be trusted in these limits!

In fact, the potential V in this example is not smooth — while the WKB approximation, in fact, assumes a smooth potential (see remark 4.1 below). Hence can we trust the WKB answers in this case? — see next part for a (partial) answer.

Remark 4.1 ***Where does smoothness play a role in WKB theory?***
In WKB we solve the equation $\epsilon^2 y'' = Q(x)\, y$ — where $0 < \epsilon \ll 1$ — as follows[1]

First we assume a solution of the form $y(x) = A(x)\, \exp\left(\pm\frac{1}{\epsilon}\, S(x)\right)$, where $S' = \lambda = \sqrt{Q}$. Thus A must satisfy the equation

$$\pm 2\sqrt{\lambda}\left(\sqrt{\lambda}\, A\right)' + \epsilon\, A'' = 0. \tag{4.8}$$

Then we expand $A \sim \sum_{n=0}^{\infty} \epsilon^n A_n$, so that

$$\pm 2\sqrt{\lambda}\left(\sqrt{\lambda}\, A_{n+1}\right)' + A_n'' = 0, \quad \textit{for } n \geq 0, \tag{4.9}$$

where $A_0 = c/\sqrt{\lambda} = c\, Q^{-1/4}$ — with c a constant. Now, suppose that Q fails to have a second derivative (for example, Q has a corner) someplace. Then A_0 will fail to have a second derivative as well, and equation (4.9) will be undefined for $n = 0$. More generally, if Q has derivatives only up to order N (for some N), then equation (4.9) implies that (in general): A_0 has derivatives up to order N only, A_1 has derivatives up to order $N-1$ only, A_2 has derivatives up to order $N-2$ only, and so on. Clearly, at some point the expansion will fail.

When Q is piecewise smooth, we ***can get around the problem*** *by using a different WKB expansion on each side of the points where smoothness fails, and then matching the solution and its derivative across these points.* ***But, this can get rather messy!***

Solution to extra credit. As in part (b), **we assume that** $a = 1$**, with** $V = 1 - |x|$ **for** $|x| < 1$. But now we solve the equations exactly — for $\boxed{\omega <\neq 0}$ — as follows:

$$\begin{aligned}
\Phi &= \exp\left\{-i\,\frac{\omega}{\epsilon}\,x\right\} + R\,\exp\left\{+i\,\frac{\omega}{\epsilon}\,x\right\}, \quad \text{for } x < -1, && (4.10)\\
\Phi &= T\,\exp\left\{-i\,\frac{\omega}{\epsilon}\,x\right\}, \quad \text{for } x > +1, && (4.11)\\
\Phi &= c_1\,\mathrm{Ai}\left\{\frac{1+x-\omega^2}{\epsilon^{2/3}}\right\} + c_2\,\mathrm{Bi}\left\{\frac{1+x-\omega^2}{\epsilon^{2/3}}\right\}, \quad \text{for } -1 < x < 0, && (4.12)\\
\Phi &= c_3\,\mathrm{Ai}\left\{\frac{1-x-\omega^2}{\epsilon^{2/3}}\right\} + c_4\,\mathrm{Bi}\left\{\frac{1-x-\omega^2}{\epsilon^{2/3}}\right\}, \quad \text{for } 0 < x < 1, && (4.13)
\end{aligned}$$

[1] Here we consider the case $Q > 0$ only. The case $Q < 0$ is similar.

where c_1, c_2, c_3, and c_4 are constants — determined by the condition that Φ and its derivative must be continuous. The **continuity of Φ and Φ', enforced at $x = \pm 1$ and $x = 0$, leads to six linear equations (that uniquely determine R, T and the four c_j's — in terms of ω) as follows:**

Continuity of Φ and Φ' at $x = 1$. This leads to two conditions, that we state in the vector form:

$$\mathbf{Q}\left(-\zeta^2\right)\vec{c}_p = \overline{\lambda}\,T\,\vec{r} \quad \Longrightarrow \quad \vec{c}_p = \overline{\lambda}\,T\,\mathbf{P}\left(-\zeta^2\right)\vec{r}, \tag{4.14}$$

where the overbar denotes the complex conjugate, $\boxed{\zeta = \frac{\omega}{\epsilon^{1/3}}, \;\; \lambda = \exp\left(i\,\frac{\omega}{\epsilon}\right),}$

$$\mathbf{Q}(z) = \begin{bmatrix} \mathrm{Ai}(z) & \mathrm{Bi}(z) \\ \mathrm{Ai}'(z) & \mathrm{Bi}'(z) \end{bmatrix}, \quad \vec{c}_p = \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}, \quad \vec{r} = \begin{bmatrix} 1 \\ i\,\zeta \end{bmatrix}, \quad \text{and} \quad \mathbf{P}(z) = \pi \begin{bmatrix} \mathrm{Bi}'(z) & -\mathrm{Bi}(z) \\ -\mathrm{Ai}'(z) & \mathrm{Ai}(z) \end{bmatrix}.$$

Note that $\mathbf{P} = \mathbf{Q}^{-1}$, as follows **from the Wronskian** identity $\mathrm{Ai}\,\mathrm{Bi}' - \mathrm{Ai}'\,\mathrm{Bi} = 1/\pi$.

Continuity of Φ and Φ' at $x = 0$. This leads to two conditions, that we state in the vector form:

$$\mathbf{Q}\,(\xi)\;\vec{c}_l = \mathbf{M}\,(\xi)\;\vec{c}_p \quad \Longrightarrow \quad \vec{c}_l = \mathbf{P}\,(\xi)\;\mathbf{M}\,(\xi)\;\vec{c}_p \tag{4.15}$$

where $\boxed{\xi = \frac{1-\omega^2}{\epsilon^{2/3}},}$ $\quad \vec{c}_l = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $\mathbf{M}(z) = \begin{bmatrix} \mathrm{Ai}(z) & \mathrm{Bi}(z) \\ -\mathrm{Ai}'(z) & -\mathrm{Bi}'(z) \end{bmatrix}$

Continuity of Φ and Φ' at $x = -1$. This leads to two conditions, that we state in the vector form:

$$\begin{bmatrix} \lambda & \overline{\lambda} \\ -i\,\zeta\,\lambda & i\,\zeta\,\overline{\lambda} \end{bmatrix}\begin{bmatrix} 1 \\ R \end{bmatrix} = \mathbf{Q}\left(-\zeta^2\right)\vec{c}_l \quad \Longrightarrow \quad \begin{bmatrix} 1 \\ R \end{bmatrix} = \underbrace{\frac{1}{2\,i\,\zeta}\begin{bmatrix} i\,\zeta\,\overline{\lambda} & -\overline{\lambda} \\ i\,\zeta\,\lambda & \lambda \end{bmatrix}}_{\mathbf{N}}.\mathbf{Q}\left(-\zeta^2\right)\vec{c}_l \tag{4.16}$$

Putting all three continuity conditions together:

$$T^{-1}\begin{bmatrix} 1 \\ R \end{bmatrix} = \overline{\lambda}\,\mathbf{N}\,\mathbf{Q}\left(-\zeta^2\right)\,\mathbf{P}\,(\xi)\;\mathbf{M}\,(\xi)\;\mathbf{P}\left(-\zeta^2\right)\vec{r}, \tag{4.17}$$

which gives an **"explicit" expression for R and T in terms of ω and ϵ.**

Notice now that, as long as $\omega \gg \epsilon^{1/3}$, and $|\omega - 1| \gg \epsilon^{2/3}$, we can use the asymptotic formulas for Ai and Bi — for large values of its arguments — in equation (4.17), to obtain asymptotic expressions for R and T in the $0 < \epsilon \ll 1$ limit. **This, however, is easier said than done, and we will only do it partially — as the calculations involved are quite messy.**

Consider the case when $0 < \omega < 1$. Then we can argue that c_2 **in equation (4.12) has to be exponentially small.** Otherwise the term involving Bi would become dominant (and very large) by $x = 0$, which would eventually lead to a very large T — not possible, since it can be shown that $|R^2| + |T^2| = 1$. Notice that this is, essentially, the same argument made in the WKB calculation, when using the connection formulas across the left turning point.

> **Sketch of the proof that $|R^2| + |T^2| = 1$:** Consider Φ and its complex conjugate $\overline{\Phi}$. Then both are solutions of the equation $\epsilon^2\, y'' = (V - \omega^2)\, y$, and their Wronskian $W = \Phi\, \overline{\Phi}' - \Phi'\, \overline{\Phi}$ is constant. Evaluation of W at $\pm\infty$ then yields the desired result.

We now use the fact that c_2 is exponentially small to write the conditions in equation (4.16) in the form

$$\lambda + \overline{\lambda}\, R \approx c_1\, \mathrm{Ai} \quad \text{and} \quad -\lambda + \overline{\lambda}\, R \approx \frac{1}{i\,\zeta}\, c_1\, \mathrm{Ai}', \tag{4.18}$$

where **Ai and its derivative are evaluated at** $\boxed{-\zeta^2 = -\omega^2\, \epsilon^{-2/3}}$ — and the error is exponentially small. Hence, **to within exponentially small errors, we can write:**

$$R = \frac{i\,\zeta\, \mathrm{Ai} + \mathrm{Ai}'}{i\,\zeta\, \mathrm{Ai} - \mathrm{Ai}'}\, \lambda^2, \tag{4.19}$$

where $\boxed{\lambda = \exp(i\,\omega/\epsilon).}$ Use in this last equation of the large argument asymptotic expansions for the Airy function yields, at leading order, the same answer for R as in equation (4.7). However, were we to compute the **higher order corrections from WKB theory to R in equation (4.7), we would find that they differ from the higher order corrections provided by (4.19) above.**

The case $\omega > 1$ provides a striking illustration of the differences that the introduction of a non-smooth potential introduces. In this situation, for a smooth potential, WKB theory predicts a wave reflection coefficient R which is smaller than any power of ϵ for $0 < \epsilon \ll 1$. That this is not the case here is shown by figure 4.1, where the absolute value of R — as provided by a numerical evaluation of equation (4.17), is plotted as a function of ϵ, for a fixed value of $\omega = 1.5$.

In general, one can show that the rate of decay of R as a function of ϵ, as in the situation being considered here, is directly related to the degree of smoothness of the potential. In particular, each

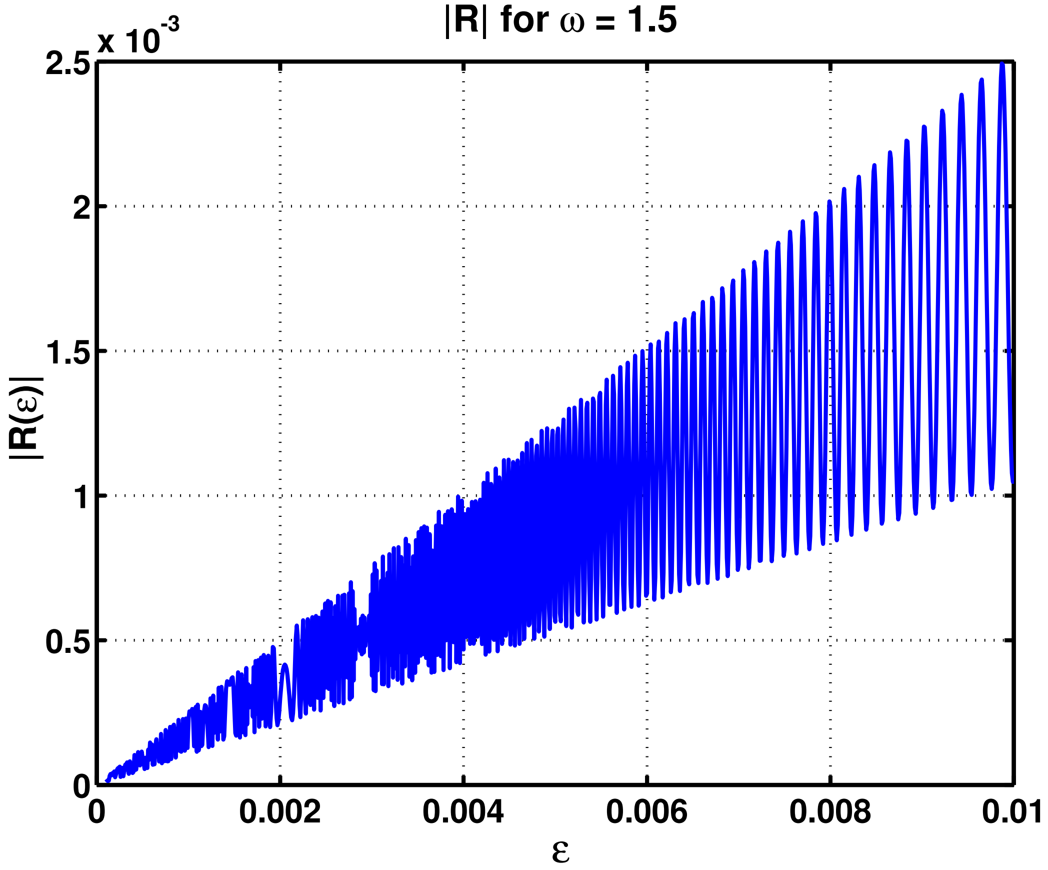

Figure 4.1: Plot of the absolute value of the reflection coefficient, as a function of ϵ, for an ω such that $\omega^2 > \max(V)$. Were V a smooth potential, R would be smaller than any power of ϵ for such an ω. This is clearly not the case here, where we see that $R = O(\epsilon)$. This behavior is produced because the potential has corners: its derivative has discontinuities.

extra derivative provides an extra power of ϵ. The situation is, in fact, very similar to that of the decay rate of the Fourier Transform for large values of k.

THE END.