Problem Set # 3, 18.305. MIT (Fall 2005)

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November 10, 2005.

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1 Airy functions integrals (statement).

Consider three paths in the complex plane with the following properties:

- Γ_1 goes from $|z| = \infty$ along the radial line $\arg(z) = (5/6)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (1/6)\pi$.
- Γ_2 goes from $|z| = \infty$ along the radial line $\arg(z) = (3/2)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (1/6)\pi$.
- Γ_3 goes from $|z| = \infty$ along the radial line $\arg(z) = (5/6)\pi$ to $|z| = \infty$ along the radial line $\arg(z) = (3/2)\pi$.

In the lectures it was shown that

$$Ai(x) = \frac{1}{2\pi} \int_{\Gamma_1} e^{i(xz + \frac{1}{3}z^3)}\, dz \quad \text{and} \quad Bi(x) = \frac{1}{2\pi i} \int_{\Gamma_2} e^{i(xz+\frac{1}{3}z^3)}\, dz + \text{c.c.}. \tag{1.1}$$

are both solutions of the Airy equation $y'' = xy$, where c.c. denotes the complex conjugate.

Show that the above are equivalent to:

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(xt + \frac{1}{3}t^3\right) dt, \tag{1.2}$$

and

$$Bi(x) = \frac{1}{\pi} \int_0^{\infty} \sin\left(xt + \frac{1}{3}t^3\right) dt \quad + \quad \frac{1}{\pi} \int_0^{\infty} \exp\left(xt - \frac{1}{3}t^3\right) dt. \tag{1.3}$$

Solution to the Airy functions integrals problem.

We wish to express the given integrals in terms of real functions, so we investigate how we can deform the path of integration, preserving the convergence of the integral as well. For fixed x, we have convergence when the contour is deformed in regions of the complex plane that satisfy

$$\Re\left\{iz^3\right\} = -r^3 \sin(3\,\theta) \le 0 \quad \text{as} \quad |z| = r \to \infty, \quad \text{where} \;\; \theta = \arg(z)\,.$$

The convergence regions are thus

$$0 \le \theta \le \pi/3, \quad 2\pi/3 \le \theta \le \pi, \quad \text{and} \quad 4\pi/3 \le \theta \le 5\pi/3.$$

These are the unshaded regions in figure 1.1. By deforming the contour within these regions, we are guaranteed that the contributions from the "circular" arcs in the path go to zero at infinity. Guided by this figure, we deform Γ_1 to the real line, so that

$$\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{2\pi} \int_{\Gamma_1} e^{i\left(xz+z^3/3\right)}\, dz \\
&= \frac{1}{2\pi} \left(\int_0^{+\infty} e^{i\left(xt+t^3/3\right)}\, dt + \int_{-\infty}^{0} e^{i\left(xt+t^3/3\right)}\, dt \right) \\
&= \frac{1}{2\pi} \int_0^{+\infty} \left(e^{i\left(xt+t^3/3\right)} + e^{-i\left(xt+t^3/3\right)} \right) dt \\
&= \frac{1}{\pi} \int_0^{+\infty} \cos\left(xt + t^3/3\right)\, dt.
\end{aligned}$$

Figure 1.1:

Contour Deformation for the Airy integrals.

Likewise, Γ_2 can be deformed into the path consisting of the negative imaginary axis ($z = it$, with $-\infty < t < 0$), followed by the positive real axis ($z = t$, with $0 < t < \infty$) — see figure 1.1. This gives

$$\begin{aligned}
\mathrm{Bi}\,(x) &= \frac{1}{2\pi i}\int_{\Gamma_2} e^{i\left(xz+z^3/3\right)}\,dz + c.c. \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-\left(xt-t^3/3\right)}\,dt + \frac{1}{i}\int_{0}^{+\infty} e^{i\left(xt+t^3/3\right)}\,dt\right) + c.c. \\
&= \frac{1}{\pi}\int_{0}^{+\infty} e^{xt-t^3/3}\,dt + \frac{1}{\pi}\int_{0}^{+\infty} \sin\left(xt+t^3/3\right)\,dt,
\end{aligned}$$

where c.c. denotes the complex conjugate.

2 Airy functions expansions (statement).

In the lectures we showed that the exponentially decaying solutions for the Airy equation $\epsilon^2\, y'' = xy$ (for $x > 0$ and $0 < \epsilon \ll 1$), had an asymptotic expansion of the form

$$y \sim x^{-1/4}\left(\sum_{0}^{\infty} a_n\,(\epsilon/\zeta)^n\right) e^{-\zeta/\epsilon} \quad \text{where} \quad \zeta = \frac{2}{3}\,x^{3/2}. \tag{2.1}$$

Part (a) Calculate the coefficients a_n, assuming that $a_0 = 1$.

For $x < 0$ the solutions admit asymptotic expansions of the form

$$y \sim |x|^{-1/4}\left(\sum_{0}^{\infty} d_n\,(\epsilon/\eta)^n\right) e^{i\eta/\epsilon} \tag{2.2}$$

and its complex conjugate, where $\eta = \dfrac{2}{3}\,|x|^{3/2}$.

Part (b) Calculate the coefficients d_n, assuming that $d_0 = 1$.

Solution to the Airy functions expansions problem.

The solution to part(b) is the same as the solution to $\boxed{\epsilon^2 y'' = -x\,y}$ — with $x > 0$. This allows us to do just one computation and get both series, by using the WKB expansion

$$y_\pm \sim \left(\frac{3\epsilon}{2}\tau\right)^{-1/6} e^{-\beta_\pm \tau} \sum_{n=0}^{\infty} b_n^\pm \, \tau^{-n}, \quad \text{with } \tau = \frac{2}{3\epsilon}\, x^{3/2},$$

to solve $y'' = \pm xy$, where $b_n^+ = a_n$, $b_n^- = d_n$, $\beta_+ = 1$, and $\beta_- = -i$. The problem is simplified by the change of variable $x \to \tau$. Then the equation becomes

$$\frac{d^2 y}{d\tau^2} + \frac{1}{3\tau}\frac{dy}{d\tau} = \pm y.$$

Write now $\boxed{y_\pm = w_\pm(\tau)\, e^{-\beta_\pm \tau}}$ — where we expand $\boxed{w_\pm \sim \left(\frac{3\epsilon}{2}\right)^{-1/6} \sum_{n=0}^{\infty} b_n^\pm \, \tau^{-n-1/6}.}$ Then

$$2\beta_\pm \frac{dw_\pm}{d\tau} + \frac{\beta_\pm}{3\,\tau} w_\pm = \frac{d^2 w_\pm}{d\tau^2} + \frac{1}{3\,\tau}\frac{dw_\pm}{d\tau}.$$

Substituting into this equation the expansion for $w_\pm$ above, and equating equal powers of τ, we then get the following recursion relation for the coefficients $b_n^\pm$

$$b_n^\pm = -\frac{(n-1/6)(n-5/6)}{2\,n\,\beta_\pm}\, b_{n-1}^\pm.$$

Hence, since $b_0^\pm = 1$, we have

$$b_n^\pm = \left(-\frac{1}{2\,\beta_\pm}\right)^n \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{\Gamma(1/6)\,\Gamma(5/6)\,n!} = \left(-\frac{1}{2\,\beta_\pm}\right)^n \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{2\,\pi\,n!},$$

where we have used the fact that $\Gamma(1/6)\,\Gamma(5/6) = 2\,\pi$. Hence, since $a_n = b_n^+$ and $d_n = b_n^-$,

$$\boxed{a_n = \left(-\frac{1}{2}\right)^n \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{2\,\pi\,n!}} \quad \text{and} \quad \boxed{d_n = \left(-\frac{i}{2}\right)^n \frac{\Gamma(n+1/6)\,\Gamma(n+5/6)}{2\,\pi\,n!}.}$$

3 Generic 2nd order equation WKB (statement).

Let $0 < \epsilon \ll 1$, and consider the second order ODE

$$\epsilon^2\, y'' + \epsilon a\, y' + hy = 0, \tag{3.1}$$

where $a = a(x)$ and $h = h(x)$ are some given functions.

(a) Use a WKB-like approach to get asymptotic expansions for the solutions of this equation.

(b) Where do you expect the expansions obtained in the prior step to break down? Explain.

Note: The expansions will have the form $y \sim A\, e^S$, where S is appropriately selected and A has an expansion of the form $A \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \ldots$. You should write the equations satisfied by the A_n.

Solution to the Generic 2nd order equation WKB problem.

Part (a): Substituting $y = e^S$ into the equation yields $\epsilon^2 \left(S'' + (S')^2\right) + \epsilon S' a + h = 0$. Dominant balance (something has to balance the last term h in this equation) suggests that we should expand $S \sim \dfrac{1}{\epsilon}\, S_0 + S_1 + \epsilon\, S_2 + \ldots$ Alternatively we may write

$$y = A\, e^{S_0/\epsilon}, \tag{3.2}$$

with $A = e^{S - S_0/\epsilon} \sim A_0 + \epsilon\, A_1 + \epsilon^2\, A_2 + \ldots$ Define $\lambda = S_0'$ and substitute (3.2) into (3.1). Thus

$$\left(\lambda^2 + \lambda a + h\right)\, A + \epsilon\, \{2\, \lambda\, A' + \lambda'\, A + a\, A'\} + \epsilon^2\, A'' = 0. \tag{3.3}$$

The leading order $O(\epsilon^0)$ in this equation yields $\lambda^2 + \lambda\, a + h = 0 \quad \Rightarrow \quad \boxed{\lambda = -\frac{1}{2}\, a \pm \frac{1}{2}\, \sqrt{a^2 - 4h}.}$
Hence we get *two solutions* (as expected) *except* when

$$a^2 - 4h = (a + 2\lambda)^2 = 0. \tag{3.4}$$

The $O(\epsilon)$ terms in equation (3.3) yield an equation determining A_0

$$(a + 2\, \lambda)\, A_0' + \lambda'\, A_0 = 0, \tag{3.5}$$

while the higher order terms yield equations determining the A_n's, for $n > 0$. Namely

$$(a + 2\, \lambda)\, A_n' + \lambda'\, A_n = -A_{n-1}''. \tag{3.6}$$

We notice that these equations — i.e. (3.5 – 3.6) — recursively determine the coefficients A_n, *except* when equation (3.4) holds.

To find the leading order WKB solution, we note that

$$\frac{A_0'}{A_0} = -\frac{1}{2}\,\frac{(a+2\,\lambda)'}{a+2\,\lambda} + \frac{1}{2}\,\frac{a'}{a+2\,\lambda}.$$

Hence, using the fact that $a+2\,\lambda = \pm\sqrt{a^2-4\,h}$ — see equation (3.4), we obtain

$$A_0 = c\,|a^2-4h|^{-1/4}\,\exp\left(\pm\frac{1}{2}\int^x \frac{a'}{\sqrt{a^2-4h}}\,ds\right),$$

where c is a constant. It follows that

$$y \sim c\,|a^2-4h|^{-1/4}\,\exp\left\{\frac{1}{2\,\epsilon}\int^x \left(-a \pm \sqrt{a^2-4h} \pm \epsilon\,\frac{a'}{\sqrt{a^2-4h}}\right)\,ds\right\}.$$

Part (b): Notice that the A_n's become successively more singular near the points where $a+2\,\lambda$ vanishes. Hence, from (3.4), the expansion breaks down at the points where

$$\boxed{a^2-4h=0.}$$

At the points where a^2-4h has a simple zero, the solution switches from oscillatory on one side of the zero, to exponential on the other side. The Airy equation gives an example of this for a zero at the origin — $a=0$ and $h=x$.

4 WKB expansion for a 3-rd order equation (statement).

Let $0 < \epsilon \ll 1$, and consider the 3-rd order ODE

$$\epsilon^3\,\frac{d^3y}{dx^3} + V\,y = 0, \tag{4.1}$$

where $V = V(x)$ is some given function.

(a) Use a WKB-like approach to get asymptotic expansions for the solutions of this equation.

(b) Where do you expect the expansions obtained in the prior step to break down? Explain.

(c) Consider the case $V = -x$, and $x > 0$. What is the condition for validity of these expansions?

Note: The expansions will have the form $y \sim A\, e^S$, where S is appropriately selected and A has an expansion of the form $A \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \ldots$. For part (a) you should provide expressions for S and (at least) the leading order term A_0. For part (c) the A_n will be powers of x, and your answer should take the form that the expansions are valid provided $\epsilon^\mu \ll x$, for some μ.

Solution of the WKB expansion for a 3-rd order equation.

Part (a): As usual, the substitution $\boxed{y = e^S}$ — and use of dominant balance — leads to the ansatz $\boxed{y = A\, e^{S_0/\epsilon},}$ with $\boxed{S_0' = \lambda,}$ and $\boxed{\lambda^3 = -V.}$ This gives **three solutions, except at points where V vanishes.** The equation for A is then

$$\underbrace{3\,\lambda^2\,A' + 3\,\lambda\,\lambda'\,A}_{3\,\lambda\,(\lambda\,A)'} + \epsilon\,(\lambda''\,A + 3\,\lambda'\,A' + 3\,\lambda\,A'') + \epsilon^2\,A''' = 0.$$

Hence expanding $A = A_0 + \epsilon\,A_1 + \ldots$ we get (for some arbitrary constant c)

$$3\,\lambda\,(\lambda\,A_0)' = 0 \qquad \Rightarrow \qquad A_0 = c/\lambda. \tag{4.2}$$

Likewise, A_1 satisfies

$$3\,\lambda\,(\lambda\,A_1)' + \lambda''\,A_0 + 3\,\lambda'\,A_0' + 3\,\lambda\,A_0'' = 0.$$

In general, for $n > 1$

$$3\,\lambda\,(\lambda\,A_n)' + \left(\lambda''\,A_{n-1} + 3\,\lambda'\,A_{n-1}' + 3\,\lambda\,A_{n-1}''\right) + A_{n-2}''' = 0.$$

Part (b): It should be clear from (4.2) that **the expansion fails when $\lambda = 0$ — i.e. when V has a zero.** At these points the expansion predicts that the solutions to the equation reach infinity. This, obviously, does not happen — since the solutions to the equation are perfectly regular near points where V vanishes.

Part (c): Let $V = -x$ with $x > 0$. Then $\lambda^3 = x$ — i.e. $\boxed{\lambda = \sigma\, x^{1/3}, \;\; \text{with} \;\; \sigma^3 = 1.}$ Thus

$$S_0 = \frac{3}{4}\,\sigma\, x^{4/3}$$

so that $\boxed{y \sim A \, \exp\left(\frac{3}{4\,\epsilon}\,\sigma\, x^{4/3}\right),}$ where A satisfies

$$3\,\sigma^2\, x^{1/3}\, \left(x^{1/3}\, A\right)' + \epsilon\, \left[3\,\sigma\, \left(x^{1/3}\, A'\right)' - \frac{2}{9}\,\sigma\, x^{-5/3}\, A\right] + \epsilon^2\, A''' = 0.$$

It is easy to see that this leads to an expansion of the form

$$A \sim \sum_{n=0}^{\infty} \epsilon^n\, a_n\, x^{-(1+4n)/3},$$

for some coefficients a_0, a_1, a_2, ... Thus the n-th term in this expansion has size

$$x^{-1/3}\, \left(\epsilon\, x^{-4/3}\right)^n.$$

Hence for each term to be smaller than the next we need

$$\epsilon\, x^{-4/3} \ll 1.$$

Thus **the condition for validity of the expansion is that** $\boxed{x \gg \epsilon^{3/4}.}$

Remark 4.1 *It is interesting to consider what happens for a more general type of zero for V. Thus, assume $V = -x^\mu$ for $x > 0$, for some $\mu > 0$. It is easy to see that this leads to a solution of the asymptotic expansion of the form*

$$A \sim \sum_{n=0}^{\infty} \epsilon^n\, a_n\, x^{-\frac{1}{3}\,\mu - n\,\alpha},$$

where $\alpha = 1 + \dfrac{1}{3}\,\mu$, and the a_n's are some coefficients. This is then valid provided that $\boxed{x \gg \epsilon^{1/\alpha}.}$

THE END.