Answers to Problem Set Number 2
for 18.305 — MIT (Fall 2005)

D. Margetis and R. Rosales (MIT, Math. Dept., Cambridge, MA 02139).

October 24, 2005.

Course TA: Nikos Savva, MIT, Dept. of Mathematics, Cambridge, MA 02139.

Contents

List of Figures

1 Problem 1

Find the leading term for each of the integrals below using Laplace method for $\lambda \gg 1$.

$$\textbf{(a)} \int_{-2}^{1} dx\ e^{-\lambda x^3} \left(1 + x^2\right), \quad \textbf{(b)} \int_{1}^{\infty} dx\ e^{-\lambda \cosh x} \sqrt{x^2 - 1}, \quad \text{and} \quad \textbf{(c)} \int_{0}^{1} dx\ e^{\lambda x^2 (1-x)}.$$

Solution to problem 1.

(a) Let Let $\boxed{I_1 = \int_{-2}^{1} dx\; e^{-\lambda x^3}\left(1+x^2\right).}$

The biggest contribution comes from the neighborhood of $x = -2$. Therefore we get

$$\begin{aligned} I_1 &\sim \int_{-2}^{1} dx\; e^{-\lambda(-8+12(x+2)+\ldots)}\left(1+x^2\right) && \text{Expand } x^3 \text{ about } x=-2 \\ &\sim \int_{0}^{3} dy\; e^{-\lambda(-8+12y)}\left(1+(y-2)^2\right) && \text{Change of Variable } y = x+2 \\ &\sim 5e^{8\lambda}\int_{0}^{+\infty} dy\; e^{-12\lambda y} \end{aligned}$$

To leading order, the integral is thus .... $\boxed{I_1 \sim \frac{5}{12}\frac{e^{8\lambda}}{\lambda}.}$

(b) Let $\boxed{I_2 = \int_{1}^{+\infty} dx\; e^{-\lambda\cosh x}\sqrt{x^2-1}.}$

Then the neighborhood of $x = 1$ contributes the most to the integral and we get:

$$\begin{aligned} I_2 &\sim \int_{1}^{+\infty} dx\; \sqrt{(x-1)(x+1)}\,e^{-\lambda[\cosh 1+(x-1)\sinh 1+\ldots]} && \text{Expand } \cosh \text{ about } x=1 \\ &\sim e^{-\lambda\cosh 1}\int_{0}^{+\infty} dy\; \sqrt{y(y+2)}\,e^{-\lambda y\sinh 1} && \text{Change of Variable } y = x-1 \\ &\sim e^{-\lambda\cosh 1}\int_{0}^{+\infty} dy\; \sqrt{2y}\left(1+\frac{y}{4}+\ldots\right)e^{-\lambda y\sinh 1} && \text{Expand } \sqrt{y(y+2)} \\ &\sim \sqrt{2}e^{-\lambda\cosh 1}\int_{0}^{+\infty} dy\; y^{1/2}e^{-\lambda y\sinh 1} \\ &\sim \frac{\sqrt{2}e^{-\lambda\cosh 1}}{(\lambda\sinh 1)^{3/2}}\int_{0}^{+\infty} dw\; w^{1/2}e^{-w} && \text{Change of variable } w = \lambda y\sinh 1 \\ &\sim \frac{\sqrt{2}\,\Gamma(3/2)\,e^{-\lambda\cosh 1}}{(\lambda\sinh 1)^{3/2}} \end{aligned}$$

To leading order, the integral is thus .... $\boxed{I_2 \sim \frac{\sqrt{\pi}e^{-\lambda\cosh 1}}{\sqrt{2}\,(\lambda\sinh 1)^{3/2}}.}$

(c) Let $\boxed{I_3 = \int_0^1 dx\ e^{\lambda x^2(1-x)}.}$

Here we maximize the contribution to the integral in the neighborhood of $x = \bar{x}$ which satisfies

$$\frac{d}{dx}\left(\bar{x}^2\,(1-\bar{x})\right) = 0 \Rightarrow \bar{x} = \frac{2}{3}$$

Then

$$\begin{aligned} I_3 &\sim \int_0^1 dx\ e^{\lambda\left[4/27-(x-2/3)^2+\ldots\right]} && \text{Expand } x^2\,(1-x) \text{ about } x = 2/3 \\ &\sim e^{4\lambda/27}\int_0^1 dx\ e^{-\lambda(x-2/3)^2} \\ &\sim e^{4\lambda/27}\int_{-2/3}^{1/3} dy\ e^{-\lambda y^2} && \text{Change of Variable } y = x - 2/3 \\ &\sim e^{4\lambda/27}\int_{-\infty}^{+\infty} dy\ e^{-\lambda y^2} && \text{Neglect end point contribution} \Rightarrow \text{send limits to } \infty \end{aligned}$$

To leading order, the integral is thus .... $\boxed{I_3 \sim e^{4\lambda/27}\sqrt{\frac{\pi}{\lambda}}.}$

2 Problem 2

Find the leading term for each of the integrals below using Laplace method for $\lambda \gg 1$.

$$\textbf{(a)}\int_{-\pi/2}^{\pi/2} dx\ e^{-\lambda\cos x} \quad \text{and} \quad \textbf{(b)}\int_0^{\infty} dx\ e^{-\lambda\left(x+x^5\right)}.$$

Solution to Problem 2.

(a) We have

$$I_1 = \int_{-\pi/2}^{\pi/2} dx\ e^{-\lambda\cos x} = 2\int_0^{\pi/2} dx\ e^{-\lambda\cos x} \tag{2.1}$$

The neighborhood of $x = \pi/2$ contributes the most to the integral. Expanding $\cos x$ about $x = \pi/2$ gives:

$$I_1 \sim 2\int_0^{\pi/2} dx\ e^{\lambda[(x-\pi/2)+\ldots]}$$

To leading order, the integral is thus .... $\boxed{I_1 \sim \frac{2}{\lambda}.}$

(b) Let

$$I_2 = \int_0^{+\infty} dx\ e^{-\lambda\left(x+x^5\right)} \tag{2.2}$$

The minimum of $x + x^5$ is attained at $x = 0$. To leading order we get

$$I_2 \sim \int_0^{+\infty} dx\ e^{-\lambda x} \quad \Longrightarrow \quad \boxed{I_2 \sim \tfrac{1}{\lambda}.} \tag{2.3}$$

3 Problem 3

Find the *entire* asymptotic series for the integrals $(\lambda \gg 1)$

$$\textbf{(a)} \int_{-\pi/2}^{\pi/2} dx\ e^{-\lambda \cos x} \quad \text{and} \quad \textbf{(b)} \int_0^{\infty} dx\ e^{-\lambda\left(x+x^5\right)}.$$

Solution to Problem 3.

(a) Make the change of variable $\rho = \lambda \cos x$. Then equation (2.1) becomes:

$$I_1 = 2 \int_0^{\lambda} d\rho\ \frac{e^{-\rho}}{\lambda} \left(1 - \left(\frac{\rho}{\lambda}\right)^2\right)^{-1/2}$$

From the lecture notes we have that for $|\xi| < 1$,

$$(1-\xi)^{-1/2} = \sum_{m=0}^{+\infty} \frac{\Gamma\left(m+1/2\right)}{m!\sqrt{\pi}} \xi^m$$

Using this expansion and interchanging the integral and summation we get

$$I_1 \sim \frac{2}{\lambda} \sum_{m=0}^{+\infty} \frac{\Gamma\left(m+1/2\right)}{m!\sqrt{\pi}} \frac{1}{\lambda^{2m}} \underbrace{\int_0^{\lambda} d\rho\ e^{-\rho} \rho^{2m}}_{\Gamma(2m+1)}$$

and therefore

$$\boxed{I_1 \sim \frac{2}{\lambda\sqrt{\pi}} \sum_{m=0}^{+\infty} \frac{\Gamma\left(m+1/2\right)\Gamma\left(2m+1\right)}{m!} \frac{1}{\lambda^{2m}}}$$

(b) Make the change of variable $\rho = \lambda x$. Then equation (2.2) becomes:

$$I_2 = \frac{1}{\lambda} \int_0^{+\infty} d\rho\ e^{-\rho} e^{-\rho^5/\lambda^4} \tag{3.1}$$

Expanding $e^{-\rho^5/\lambda^4}$ when $\rho << \lambda$ gives

$$e^{-\rho^5/\lambda^4} = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \frac{\rho^{5m}}{\lambda^{4m}}$$

Plugging back into (3.1) we get

$$I_2 \sim \frac{1}{\lambda} \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!} \frac{1}{\lambda^{4m}} \underbrace{\int_0^{+\infty} d\rho \; e^{-\rho} \rho^{5m}}_{=\Gamma(5m+1)=(5m)!}$$

So finally we get:

$$\boxed{I_2 \sim \frac{1}{\lambda} \sum_{m=0}^{+\infty} \frac{(-1)^m (5m)!}{m!} \frac{1}{\lambda^{4m}}}$$

Note that both of the series expansions in this problem answer diverge. This is typical for asymptotic series. Given $\lambda \gg 1$, we must properly truncate the series to get an accurate approximation to the value of the integrals.

The divergence of asymptotic series expansions obtained from integrals (generally) arises because the order of integral signs and infinite summation signs are exchanged. Even if the infinite sums converge, upon exchange convergence will be destroyed.

The reason that asymptoticity is not destroyed by exchanging the order of summation and integration is that asymptotic infinite sums are not a single statement, but infinitely many. What they say is that: whenever the infinite sum is truncated at a finite point, then the error is of the same order as the next term in the sum. Since each of these statements involves only a finite sum, exchanging integration and summation orders is (generally) allowed.

4 Problem 4

Find the leading term for each of the integrals below using the stationary phase method for $\lambda \gg 1$.

(a) $\int_{-a}^{a} dt \; e^{i\lambda t^5}$ for $a > 0$, **(b)** $\int_{1}^{\infty} dt \; e^{i\lambda t^4} \sqrt{1+t^2}$, and **(c)** $\int_0^{\pi} dt \; e^{i\lambda \sin t}$.

Solution to Problem 4.

(a) We have $I_1 = \int_{-a}^{a} dt\ e^{i\lambda t^5} = 2\Re\left\{\int_0^a dt\ e^{-i\lambda t^5}\right\}$. Then, since $t = 0$ is the stationary phase point, we may neglect the contributions from the endpoint $t = a$. Thus we write:

$$I_1 \sim 2\Re\left\{\int_0^{+\infty} dt\ e^{i\lambda t^5}\right\}$$

We wish to make the change of variable $\boxed{t = i^{1/5} z^{1/5}}$ — with $0 < z < \infty$. This amounts to a change in the path of integration in the complex plane. Since there are five different values for $i^{1/5}$ — namely $e^{\pi i/10}$, $e^{\pi i/2}$, $e^{9\pi i/10}$, $e^{13\pi i/10}$ and $e^{17\pi i/10}$ — we need to choose the new contour so that the integral is convergent and the change in paths is allowed. To decide which path to choose we look at the integrand:

The integral converges when .... $\Re\left\{i\lambda t^5\right\} < 0$ as $|t| \to \infty$.

With $t = re^{i\theta}$ this condition is equivalently to .... $\lambda r^5 \sin 5\theta > 0$ as $|t| \to \infty$.

Thus we require .... $\boxed{\sin 5\theta > 0.}$

The shaded regions in figure 4.1 show where in the complex t-plane this last equation holds.

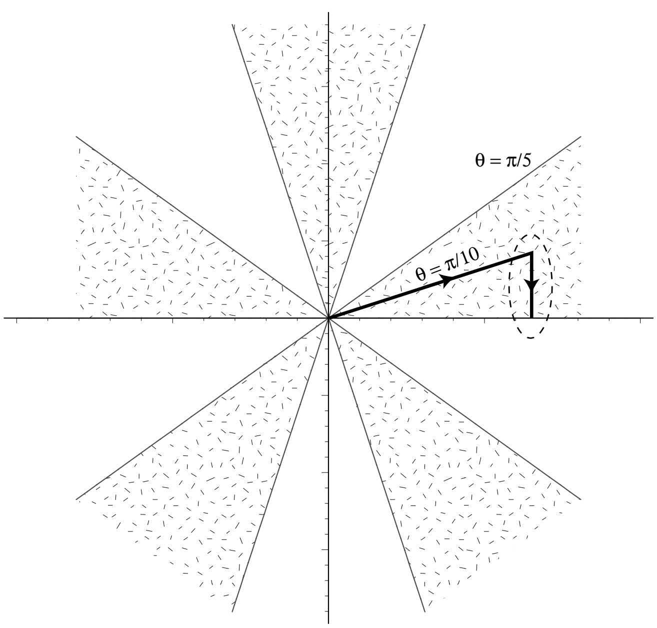

Figure 4.1:
Contour Deformation for Problem 4a. The shaded regions indicate regions where the integrand goes to zero at infinity

Thus we deform the original path of integration to the one shown in the figure. The contributions from the circled path can be dropped, since the integral over this part goes to zero at infinity. Effectively, this amounts to the change of variable $t = e^{\pi i/10} z^{1/5}$, and the integral becomes

$$I_1 \sim 2\Re\left\{e^{\pi i/10}\int_0^{+\infty} dz\, \frac{e^{-\lambda z}}{5} z^{-4/5}\right\}$$

A final change of variable $\lambda z = w$ gives .... $I_1 \sim 2\Re\left\{\frac{e^{\pi i/10}}{5\lambda^{1/5}}\int_0^{+\infty} dw\, e^{-w} w^{-4/5}\right\}$.

Thus

$$I_1 \sim 2\Re\left\{\frac{e^{\pi i/10}}{5\lambda^{1/5}}\Gamma(1/5)\right\} \quad\Rightarrow\quad \boxed{I_1 \sim \frac{2\cos(\pi/10)\,\Gamma(1/5)}{5\lambda^{1/5}}.}$$

Note: Contour of integration changes of the type done above are (almost) always needed when dealing with asymptotic expansions of conditionally convergent integrals — that converge because of an integrand that is highly oscillatory at infinity. While we (in general) will not present the details in as careful a manner as above, you should realize that such calculations are happening "behind the scenes". Failure to carry them properly will, almost always, lead to erroneous answers.

(b) For the integral $I_2 = \int_1^{+\infty} dt\, e^{i\lambda t^4}\sqrt{1+t^2}$ the dominant contribution comes from near $t = 1$.

Expanding t^4 near $t = 1$ gives .... $I_2 \sim \int_1^{+\infty} dt\, e^{i\lambda(1+4(t-1)+\dots)}\sqrt{1+t^2}$.

The change of variable $\rho = t - 1$ then yields

$$I_2 \sim \int_0^{+\infty} d\rho\, e^{i\lambda(1+4\rho)}\sqrt{1+(\rho+1)^2} \sim e^{i\lambda}\int_0^{+\infty} d\rho\, e^{4i\lambda\rho}\sqrt{1+(\rho+1)^2}$$

$$\sim \sqrt{2}e^{i\lambda}\int_0^{+\infty} d\rho\, e^{4i\lambda\rho} \quad\Rightarrow\quad \boxed{I_2 \sim \frac{i\sqrt{2}e^{i\lambda}}{4\lambda}.}$$

(c) The stationary phase point for $I_3 = \int_0^{\pi} dt\, e^{i\lambda\sin t}$ is at $t = \pi/2$. Expanding the $\sin t$ term in the integrand near $t = \pi/2$ gives

$$I_3 \sim \int_0^{\pi} dt\, e^{i\lambda\left(1-(t-\pi/2)^2/2\right)} \sim e^{i\lambda}\int_0^{\pi} dt\, e^{-i\lambda(t-\pi/2)^2/2}$$

$$\sim e^{i\lambda}\int_{-\pi/2}^{\pi/2} dy\, e^{-i\lambda y^2/2} \sim e^{i\lambda}\int_{-\infty}^{+\infty} dy\, e^{-i\lambda y^2/2}.$$

Next, an argument similar to the one given in part (a) indicates that the path should be deformed to satisfy $\sin 2\theta < 0$. Effectively, this means that we can do the change of variable $y = e^{-\pi i/4} z$, with z real. Then

$$I_3 \sim e^{i\lambda} e^{-\pi i/4} \int_{-\infty}^{+\infty} dy\; e^{-\lambda z^2/2} \qquad \Rightarrow \qquad \boxed{I_3 \sim e^{i(\lambda-\pi/4)} \sqrt{\frac{2\pi}{\lambda}}.}$$

5 Problem 5

(a) Find the entire asymptotic series expansion for the integral $I = \int_{-\infty}^{+\infty} e^{i\lambda \sinh^4 t}\, dt$, where $\lambda \gg 1$.

(b) Find an upper bound for the remainder (error) of the expansion in part (a) when $N \gg 1$ terms are added. Explain what value of N you would choose for maximum accuracy in evaluating I. **HINT:** you may wish to use Stirling's formula: $\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-1/2}$ as $|z| \to \infty$ and $\arg(z) < \pi$.

Solution to Problem 5.

We will use the fact that .... $I = \int_{-\infty}^{+\infty} dt\; e^{i\lambda \sinh^4 t} = 2 \int_0^{+\infty} dt\; e^{i\lambda \sinh^4 t}$.

(a) We let $\rho = \lambda \sinh^4 t$. Under this change of variable the integral becomes

$$I = \frac{1}{2\lambda} \int_0^{+\infty} d\rho\; \left(1 + \sqrt{\frac{\rho}{\lambda}}\right)^{-1/2} \left(\frac{\rho}{\lambda}\right)^{-3/4} e^{i\rho}.$$

Next we change the path of integration in the complex plane, to exploit the exponential nature of the integrand. Note that the factor multiplying the exponential decays like $|\rho|^{-1}$ as $|\rho| \to \infty$ in the complex plane. Hence, using Jordan's lemma,[1] we can change the path of integration from the positive real axis to the positive imaginary axis, where $\rho = it$ $(0 < t < \infty)$. Thus

$$I = \frac{e^{i\pi/8}}{2\lambda} \int_0^{+\infty} dt\; \left(1 + \mu \sqrt{\frac{t}{\lambda}}\right)^{-1/2} \left(\frac{t}{\lambda}\right)^{-3/4} e^{-t}, \quad \text{where} \quad \mu = e^{i\pi/4}.$$

Now we use the binomial expansion for the square root term in the integrand, namely:

$$\left(1 + \mu \sqrt{\frac{t}{\lambda}}\right)^{-1/2} = \sum_{m=0}^{+\infty} \frac{(-\mu)^m\, \Gamma(m+1/2)}{m!\, \sqrt{\pi}} \left(\frac{t}{\lambda}\right)^{m/2},$$

---

[1] For C_R a circular arc of radius R in $\mathrm{Im}(\rho) \geq 0$, $\int_{C_R} \left(1 + \sqrt{\rho/\lambda}\right)^{-1/2} (\rho/\lambda)^{-3/4}\, e^{i\rho}\, d\rho \to 0$ as $R \to \infty$.

to obtain (after exchanging the summation and integration order):

$$I \sim \frac{e^{i\pi/8}}{2\sqrt{\pi}} \sum_{m=0}^{+\infty} \frac{(-\mu)^m\, \Gamma(m+1/2)}{m!\, \lambda^{m/2+1/4}} \underbrace{\int_0^{+\infty} dt\, t^{m/2-3/4}\, e^{-t}}_{\Gamma(m/2+1/4)}. \tag{5.1}$$

Therefore, the expansion is:

$$\boxed{I \sim \frac{e^{i\pi/8}}{2\sqrt{\pi}} \sum_{m=0}^{+\infty} \frac{(-\mu)^m\, \Gamma(m+1/2)\, \Gamma(m/2+1/4)}{m!\, \lambda^{m/2+1/4}}.} \tag{5.2}$$

IMPORTANT: *Notice that we must change the integration path BEFORE doing the exchange of the integration and summation order. If we do not, then (at the level of equation (5.1)) integrals like* $\int_0^{\infty} d\rho\, \rho^{m/2-3/4}\, e^{i\rho}$ *appear in the calculation — and such integrals do not make any sense for* $m > 1$*, as they are not convergent, not even conditionally!* This is an error MANY in the class made (including the TA, and the lecturer when checking the TA's answers).

(b) The m-th term in the asymptotic series above is

$$\alpha_m = \frac{e^{i\pi/8}}{2\sqrt{\pi}} \frac{(-\mu)^m\, \Gamma(m+1/2)\, \Gamma(m/2+1/4)}{m!\, \lambda^{m/2+1/4}}. \tag{5.3}$$

Using Stirling's formula, we notice that we can write

$$\alpha_m \sim (-\mu)^m\, e^{i\pi/8}\, 2^{7/2}\, m^{-3/4}\, \lambda^{-1/4} \left(\frac{m}{2\,\lambda\, e}\right)^{m/2} \quad \text{for} \quad m \gg 1, \tag{5.4}$$

from which it follows that the **series is not convergent, only asymptotic.** In particular, notice that

$$\left|\frac{\alpha_m}{\alpha_{m-1}}\right| \sim \sqrt{\frac{m}{2\,\lambda}} \quad \text{for} \quad m \gg 1. \tag{5.5}$$

If we now **sum the first** N **terms** ($0 \le m < N$) in the series in (5.2), a good **estimate of the error** ε_N **in the approximation is given by the first neglected term**, namely

$$\varepsilon_N \approx \alpha_N \quad \text{with} \quad |\varepsilon_N| \sim 2^{7/2}\, N^{-3/4}\, \lambda^{-1/4} \left(\frac{N}{2\,\lambda\, e}\right)^{N/2} \quad \text{for} \quad N \gg 1. \tag{5.6}$$

We notice that, for a fixed $\lambda \gg 1$, the error first decreases in size, and then increases. The "best" approximation is thus obtained for the value of N that minimizes the size of the error. Since

$$\left|\frac{\varepsilon_N}{\varepsilon_{N-1}}\right| \approx \left|\frac{\alpha_N}{\alpha_{N-1}}\right| \sim \sqrt{\frac{N}{2\,\lambda}},$$

it follows that **the best value of** N **is given by** $\boxed{N \approx 2\,\lambda.}$

THE END.