satisfies $y'' + 2y' + y = 5xe^{-x}$ and y_2 satisfies $y'' + 2y' + y = \cosh(x)$. To solve for y_1, we see that

$$
\begin{aligned}
(D^2 + 2D + 1)y_1 &= 5xe^{-x} \\
(D+1)^2 y_1 &= 5xe^{-x} \\
y_1 &= \frac{1}{D+1}\frac{1}{D+1}5xe^{-x} \\
y_1 &= e^{-x}\frac{1}{D}\frac{1}{D}5x \\
y_1 &= \frac{5}{6}x^3e^{-x}.
\end{aligned}
$$

To solve for y_2, observe that

$$
\begin{aligned}
y_2 &= \frac{1}{D^2 + 2D + 1}\cosh(x) \\
y_2 &= \frac{1}{(D+1)^2}\left(\frac{e^x + e^{-x}}{2}\right) \\
y_2 &= \frac{e^{-x}}{2}\frac{1}{D^2}\left(e^{2x} + 1\right) \\
y_2 &= \frac{1}{8}e^x + \frac{1}{4}x^2e^{-x}.
\end{aligned}
$$

We conclude by combining our results:

$$
y = y_p + y_h = \frac{5}{6}x^3e^{-x} + \frac{1}{8}e^x + \frac{1}{4}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}.
$$

18.305 Problem Set 1

Mason Rogers

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1.1a

We aim to solve $x(1+x)y' + \frac{1+x}{2}y = \sqrt{x}$, which is a first-order linear equation. First, divide through by $x(1+x)$ to obtain

$$y' + \frac{1}{2x}y = \frac{\sqrt{x}}{x(1+x)}.$$

We can multiply through by the integrating factor $\mu = \exp\left(\int \frac{1}{2x}\,dx\right) = \sqrt{x}$ to combine the left side into a product rule term and solve like so:

$$\begin{aligned}
\sqrt{x}y' + \frac{1}{2\sqrt{x}} &= \frac{1}{1+x} \\
(\sqrt{x}y)' &= \frac{1}{1+x} \\
\sqrt{x}y &= \ln(1+x) + c \\
y &= \frac{\ln(1+x)+c}{\sqrt{x}}.
\end{aligned}$$

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1.1b

The equation $y' + \ln(y) + 1 = 0$ is separable, so we separate to obtain

$$-\frac{dy}{\ln(y)+1} = dx.$$

Consider the substitution $u = -(\ln(y)+1)$, $du = -y^{-1}dy$. We have

$$\frac{e^{-u}}{u}\,du = -e\,dx.$$

Integrating both sides gives $\gamma(0,u) = -ex + c$, where γ denotes the incomplete gamma function. This can be inverted to yield a relationship for $y(x)$ as desired.

1.2b

We seek the general solution to $y'' + y' + y = \cos(x)$. To begin, we seek complementary solutions y_h to the corresponding homogeneous equation $y_h'' + y_h' + y_h = 0$. The characteristic polynomial $m^2 + m + 1$ has roots

$$m_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad m_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

giving the general complementary solution

$$y_h = e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right).$$

We now seek a particular solution to $y_p''+y_p'+y_p=\cos(x)$, or equivalently to $(D^2+D+1)y_p=\cos(x)$. We will now do some algebraic manipulations:

$$\begin{aligned}(D^2+D+1)y_p &= \cos(x)\\ y_p &= \frac{1}{D^2+D+1}\cos(x)\\ y_p &= \frac{1}{-1+D+1}\cos(x)\\ y_p &= \sin(x).\end{aligned}$$

In total, we have

$$y = y_p+y_h = \sin(x)+e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$

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1.2d

We seek the general solution to $x^3y'''+3x^2y''+3xy'+y=\ln(x)$. To begin, we seek complementary solutions y_h to the corresponding homogeneous equation $x^3y_h'''+3x^2y_h''+3xy_h'+y_h=0$. This is an Euler euqation, so solutions will be linear combinations of terms of form x^r, where r is a root of the equation $r(r-1)(r-2)+3r(r-1)+3r+1=r^3+2r+1=0$. This polynomial only has one real root, which we will denote r_1. The complex roots r_2 and r_3 are given by $a\pm bi$ for some $a,b\in\mathbb{R}$, and using the manipulation $x^{a\pm bi}=x^ax^{\pm bi}=x^ae^{\pm bi\ln(x)}$, it is clear that we can write the general complementary solution

$$y_h = c_1x^{r_1}+x^a\left(c_2\cos(b\ln(x))+c_3\sin(b\ln(x))\right).$$

We now seek a particular solution y_p to the inhomogeneous equation. Observe that for $n\geq 1$ and $a\in\mathbb{R}$, $(x^nD^n)(\ln(x)+a)=(-1)^{n-1}(n-1)!$. Thus, we hope to find a solution of form $y_p=\ln(x)+a$. Substituting this into the ODE, we obtain the constraint $a=-2$ and the solution $y_p=\ln(x)-2$. Combining with the above, we have

$$y = y_p+y_h = \ln(x)-2+c_1x^{r_1}+x^a\left(c_2\cos(b\ln(x))+c_3\sin(b\ln(x))\right).$$

1.2e

We seek the general solution to $y''+2y'+y=5xe^{-x}+\cosh(x)$. The characteristic polynomial is $(m+1)^2$, which gives the complementary solution $y_h=c_1e^{-x}+c_2xe^{-x}$. We now seek a particular solution y_p to the inhomogeneous equation. Split y_p into $y_p=y_1+y_2$, where y_1