20 /20

18.305 Problem Set 9

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1

We wish to study the asymptotic form of

$$I(\lambda) = \int_{\mathbb{R}} e^{i(\lambda t - t^3/3)}\, dt.$$

Let $f(t) = \lambda t - t^3/3$, and observe that f has saddle points where $f'(t) = \lambda - t^2 = 0$, i.e. at $t = \pm\sqrt{\lambda}$. Conveniently, both of these points are already on the real axis, our contour of integration.

Near $t = +\sqrt{\lambda}$, we have $f(\sqrt{\lambda}) = \frac{2}{3}\lambda^{3/2}$ and $f''(\sqrt{\lambda}) = -2\sqrt{\lambda}$. Accordingly, near our point of dominant contribution, we have $f(t) \approx \frac{2}{3}\lambda^{3/2} - \sqrt{\lambda}(t-\sqrt{\lambda})^2$. The contribution of this point to our integral is accordingly

$$e^{\frac{2}{3}\lambda^{3/2} i}\sqrt{\frac{\pi}{i\sqrt{\lambda}}}.$$

The contribution from the saddle point at $t = -\sqrt{\lambda}$ is the complex conjugate of this contribution, so we have

$$I(\lambda) \approx 2\mathrm{Re}\left[e^{\frac{2}{3}\lambda^{3/2} i}\sqrt{\frac{\pi}{i\sqrt{\lambda}}}\right].$$

Rewriting $i = e^{\pi i/2}$, this is seen to be

$$I(\lambda) = 2\sqrt{\pi}\lambda^{-1/4}\cos\left(\frac{2}{3}\lambda^{3/2} - \frac{\pi}{4}\right) = 2\sqrt{\pi}\left[\frac{\sin\left(\frac{2}{3}\lambda^{3/2} + \frac{\pi}{4}\right)}{\lambda^{1/4}}\right],$$

which we recognize as our usual asymptotic form of the Airy function.

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2

We wish to compute the shaded and unshaded regions of $f(z) = e^{iz^5}$. Writing $z = Re^{i\theta}$, we want to compute the shaded and unshaded regions of

$$f(z) = e^{iR^5 e^{5i\theta}} = e^{R^5 i\cos(5\theta) - R^5\sin(5\theta)}.$$

Accordingly, we have

$$|f(z)| = e^{-R^5\sin(5\theta)},$$

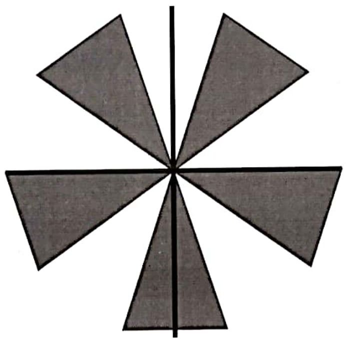

Figure 1: The shaded regions correspond to $\sin(5\theta) \leq 0$, and accordingly $f(z)$ will not decay as $|z| \to \infty$ in these regions. The unshaded regions correspond to $\sin(5\theta) > 0$, where $f(z)$ will decay as $|z| \to \infty$.

and our condition for f to decay with large R is that $\sin(5\theta) > 0$. We require $5\theta \in (0,\pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \cup (6\pi, 7\pi) \cup (8\pi, 9\pi)$, or equivalently that

$$\theta \in (0, \pi/5) \cup (2\pi/5, 3\pi/5) \cup (4\pi/5, \pi) \cup (6\pi/5, 7\pi/5) \cup (8\pi/5, 9\pi/5).$$

The complement of this region is the shaded region. We depict this in Figure 1.