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18.305 Problem Set 8

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1. We want to study the asymptotics of

$$I(\lambda) = \Gamma(\lambda+1) = \int_0^\infty e^{-t} t^\lambda \, dt.$$

We rewrite t^λ as $e^{\lambda \ln(t)}$, giving $I(\lambda) = \int_0^\infty e^{\lambda \ln(t) - t} \, dt$. The largest contribution of the integrand comes from where $\lambda \ln(t) - t$ is maximized, so we solve for its maximum. Observe that

$$\frac{d}{dt}(\lambda \ln(t) - t) = \frac{\lambda}{t} - 1,$$

which vanishes when $t = \lambda$. Accordingly, we expect our maximum contribution to occur near where $t = \lambda$. Expanding our integrand around this point up to second order derivatives gives

$$I(\lambda) \approx \int_0^\infty e^{\lambda(\ln(\lambda)-1) - \frac{1}{2\lambda}(t-\lambda)^2} \approx e^{\lambda(\ln(\lambda)-1}\sqrt{2\pi\lambda}.$$

Rearranging, we have the familiar Stirling's Approximation:

$$I(\lambda) \approx \left(\frac{\lambda}{e}\right)^\lambda \sqrt{2\pi\lambda}.$$

2. We now wish to study the asymptotics of the Laplace transform, so we have

$$I(s) = \int_0^\infty e^{-st} f(t) \, dt.$$

Suppose $f \approx f(0) + f'(0)t$ near the origin, i.e. that f is not chosen such that the dominant contribution of the integral comes from a point other than the origin. We can then evaluate the approximate integral directly, giving

$$\begin{aligned} I(s) &\approx \int_0^\infty e^{-st} f(0) \, dt + \int_0^\infty e^{-st} t f'(0) \, dt \\ &= \frac{f(0)}{s} + \frac{f'(0)}{s^2}. \end{aligned}$$

Suppose $f(t) = \cos(t)$. Then $\mathcal{L}\{f\}(s) = \frac{s}{s^2+1}$, which in the limit of large s reduces to $\frac{1}{s}$. Since $f'(0) = 0$ and $f(0) = 1$, our asymptotic formula agrees: $I(s) \approx \frac{1}{s}$.
Now suppose $f(t) = \sin(t)$. then $\mathcal{L}\{f\}(s) = \frac{1}{s^2+1}$, which reduces to $\frac{1}{s^2}$ if s is large. We now have $f'(0) = 1$ and $f(0) = 0$, so our asymptotic formula gives $I(s) \approx \frac{1}{s^2}$, which agrees.

RM: We should expand f up to its first non-vanishing derivative:

$f(t) \approx f(0) + t f'(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) = \frac{t^k}{k!} f^{(k)}(0)$, where $f(0) = \cdots = f^{(k-1)}(0) = 0$, $f^{(k)}(0) \neq 0$

Then $I(s) \approx \int_0^\infty e^{-st} \frac{t^k}{k!} f^{(k)}(0) \, dt = \frac{f^{(k)}(0)}{s^{k+1}}$

(May find answer at P_{329} of the textbook).

3. Let $f(z) = -ikz - z^4$, so $I(k) = \int_{\mathbb{R}} e^{f(z)} dz$. We look to find where the dominant contributions will come from, i.e. where f has saddle points. We find $f'(z) = -ik - 4z^3$, so $z_0^3 = \frac{-ik}{4}$. Letting $\iota = e^{2\pi i/3}$, we have

$$z_0 = i\left(\frac{k}{4}\right)^{1/3} \cdot \iota^n, \quad n \in \{0, 1, 2\}.$$

Substituting and solving for $f(z_0)$ at each of the roots, we have

$$f(z_0) = 3\left(\frac{k}{4}\right)^{4/3} \iota^n.$$

The $n = 0$ root grows large with k, so we do not expect it to be a dominant contributor to the integrand. The other two roots have negative real part, so we expect them to be dominant contributors.

While there is no contour connecting these two roots along which the phase of I is constant, the direction of steepest descent near each point is along the real axis, and the integrand is exponentially small between the two contributing roots. Thus, we deform our contour to be the line $\mathrm{Im}(z) = -\frac{3}{2}\left(\frac{k}{4}\right)^{1/3}$, passing horizontally through each of the two roots. We have

$$I(k) \approx \int_{\mathrm{Im}(z) = -\frac{3}{2}\left(\frac{k}{4}\right)^{1/3}} e^{-ikz - z^4}\, dz.$$

Near the $n = 1$ root of $f'(z)$, we have

$$f(z) \approx 3\iota\left(\frac{k}{4}^{4/3}\right) + 6\iota^2\left(\frac{k}{4}\right)^{2/3}(z - z_0)^2.$$

This point therefore contributes

$$e^{3\iota\left(\frac{k}{4}\right)^{4/3}} \sqrt{\frac{\pi}{-6\iota^2}} \left(\frac{k}{4}\right)^{-1/3}.$$

The contribution from the $n = 2$ root of $f'(z)$ gives the complex conjugate of this contribution, so we have

$$\begin{aligned} I(k) &\approx 2\mathrm{Re}\left[e^{3\iota\left(\frac{k}{4}\right)^{4/3}} \sqrt{\frac{\pi}{-6\iota^2}} \left(\frac{k}{4}\right)^{-1/3}\right] \\ &= \sqrt{\frac{2\pi}{3}} \left(\frac{k}{4}\right)^{-1/3} e^{-\frac{3}{2}\left(\frac{k}{4}\right)^{4/3}} \cos\left(\frac{3\sqrt{3}}{2}\left(\frac{k}{4}\right)^{4/3} - \frac{\pi}{6}\right). \end{aligned}$$