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18.305 Problem Set 7

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a

We want to approximate

$$I(\lambda) = \int_0^{\pi/2} e^{-\lambda \cos(t)}\, dt$$

for $\lambda \gg 1$. Observe that the integrand is maximized when $\cos(t)$ is minimized over $[0, \pi/2]$, which occurs at $t = \pi/2$. Around that point, we have $\cos(t) \sim -(t - \pi/2)$. Substituting this into the integral, we have

$$I(\lambda) \sim \int_0^{\pi/2} e^{\lambda(t-\frac{\pi}{2})}\, dt = \frac{1}{\lambda}\left(1 - e^{-\lambda\pi/2}\right) \sim \frac{1}{\lambda}.$$

Thus, to leading order, we have $I(\lambda) \sim \frac{1}{\lambda}$.

In the table, we compare with a computer result for $\lambda = 1, 10, 20, 30$:

λ	Numerical	Approximate
1	.873	1
10	.101	.1
20	.0501	.05
30	.03336	.03333

As we can see, agreement improves rather quickly as λ gets larger.

b

Now we have

$$I(\lambda) = \int_{-1}^{1} e^{\lambda t^2}\, dt = 2\int_0^1 e^{\lambda t^2}\, dt.$$

The integrand is largest when $t = 1$, so we expand around that point, writing $t^2 \sim 1 + 2(t - 1) = 2t - 1$. Substituting gives

$$I(\lambda) = 2e^{-\lambda} \int_0^1 e^{2\lambda t}\, dt = \frac{e^{-\lambda}}{\lambda}\left(e^{2\lambda} - 1\right) \sim \frac{e^{\lambda}}{\lambda}.$$

Thus, to leading order, we have $I(\lambda) \sim \frac{e^{\lambda}}{\lambda}$.

In the table, we compare with a computer result for $\lambda = 1, 10, 20, 30, 50, 70, 100, 200$:

λ	Numerical	Approximate
1	2.92	2.71
10	2336	2202
20	$2.49(10)^7$	$2.42(10)^7$
30	$3.62(10)^{11}$	$3.56(10)^{11}$
50	$1.04(10)^{20}$	$1.03(10)^{20}$
70	$3.61(10)^{28}$	$3.59(10)^{28}$
100	$2.70(10)^{44}$	$2.68(10)^{44}$
200	$3.62(10)^{84}$	$3.61(10)^{84}$

Again, we see that our approximation is quite good as λ gets larger, but convergence is much slower than in the previous case.

c

Lastly, we have

$$I(\lambda) = \int_{-\infty}^{\infty} e^{\lambda(x^2-x^4)}\, dx = 2\int_0^{\infty} e^{\lambda(x^2-x^4)}\, dx.$$

The integrand is maximized when $x^2 - x^4$ is maximized, which occurs at $x = 1/\sqrt{2}$. Around this point, we have $x^2 - x^4 \sim \frac{1}{4} - 2(x - \frac{1}{\sqrt{2}})^2$. Substituting yields

$$I(\lambda) \sim 2e^{\frac{\lambda}{4}} \int_0^{\infty} e^{-2\lambda(x-\frac{1}{\sqrt{2}})^2}\, dx = 2e^{\frac{\lambda}{4}} \int_{-1/\sqrt{2}}^{\infty} e^{-2\lambda t^2}\, dt.$$

Note that the integrand is maximized for $t = 0$ and is a factor of e smaller when $|t| \geq (2\lambda)^{-1/2}$. Thus, contributions away from zero are inconsequential, and we can extend the bottom limit of integration to $-\infty$ without affecting the result. We thus have

$$I(\lambda) = 2e^{\frac{\lambda}{4}} \int_{-\infty}^{\infty} e^{-2\lambda t^2}\, dt \approx e^{\frac{\lambda}{4}}\sqrt{\frac{2\pi}{\lambda}}.$$

In the table, we compare with a computer result for $\lambda = 1, 10, 20, 30, 50, 70, 100$:

λ	Numerical	Approximate
1	2.76	3.21
10	10.7	9.65
20	87.5	83.1
30	852	827
50	$9.67(10)^4$	$9.51(10)^4$
70	$1.20(10)^7$	$1.19(10)^7$
100	$1.81(10)^{10}$	$1.80(10)^{10}$

As we can see, agreement improves rather quickly as λ gets larger.