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18.305 Problem Set 6

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1. (a) We want to evaluate

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$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+9)(x-3i)(x-5i)},$$

which we will do using the residue theorem. Observe that the integrand has three poles: $z = 3i, z = 5i$, and $z = -3i$. Moreover, the integrand will vanish as $|z|^{-4}$ for large $|z|$, so we can add an integral about a large semicircular arc in the bottom half of the complex plane to close our contour of integration. Letting C_r be the contour given by $[-r, r]$ on the real axis and the semicircle of radius of r in the bottom half plane, we have

$$I = \lim_{r\to\infty} \oint_{C_r} \frac{dz}{(z^2+9)(z-3i)(z-5i)}.$$

We can evaluate the integral over C_r, which contains only the pole at $z = -3i$ (provided r is sufficiently large), via the residue theorem. We have

$$\mathrm{Res}_{z=-3i}(f(z)) = \frac{1}{(-6i)^2(-8i)} = \frac{1}{288i}.$$

We then have $I = -2\pi i/(288i) = -\pi/144$, where the $-2\pi i$ comes from the fact that the contour is oriented clockwise.

(b) We want to evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{(2+\sin(\theta))^2}.$$

Consider the substitution $z = e^{i\theta}$. Then we can write the integral as

$$I = \oint_C \frac{dz}{iz\left[2+\frac{1}{2i}(z+z^{-1})\right]^2} = \oint_C \frac{4iz\,dz}{(z^2+4iz-1)^2},$$

where C is the unit circle in the complex plane.
The integrand has two double poles, $z = -(2+\sqrt{3})i$ and $z = -(2-\sqrt{3})i$. Only the latter lies in the unit circle, so we aim to calculate the residue of our integrand at $z = -(2-\sqrt{3})i$. To do so, we compute

$$\mathrm{Res}_{z=-(2-\sqrt{3})i}(f(z)) = \frac{d}{dz}\left[\frac{4iz}{(z+(2+\sqrt{3})i)^2}\right]_{z=-(2-\sqrt{3})i} = \frac{2}{3\sqrt{3}i}.$$

Lastly, we multiply by $2\pi i$ per the Cauchy residue theorem, giving $I = 4\pi/(3\sqrt{3})$.

2. We have $(i\partial_t + \partial_x^2) G(x-x', t-t') = \delta(x-x')\delta(t-t')$, So

$$G(x-x', t-t') = \frac{1}{i\partial_t + \partial_x^2} \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{-ik(x-x') - ik_0(t-t')} dk_0 dk$$

Write $X = x-x'$, $T = t-t'$

$$G(X,T) = \frac{1}{i\partial_T + \partial_X^2} \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{-ikX - ik_0 T} dk_0 dk$$

The wrong way: put $\partial_T \to -ik_0$, $\partial_X^2 \to -k^2$, because then we get:

$$G(X,T) = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{-ikX - ik^2 T} \boxed{\int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{e^{-ik_0 T}}{k_0}}$$

This one is $\mathcal{F}(\frac{1}{x})$ and I can tell you

this term $= \frac{1}{2i} \operatorname{sgn}(T)$, where $\operatorname{sgn}(T) = \begin{cases} 1 & T>0 \\ -1 & T<0 \end{cases}$

Reason: For $(i\partial_t + \partial_x^2)^{-1}$, we have many choices.

Right way: $G(X,T) = \frac{1}{i\partial_T + \partial_X^2} \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{-ikX - ik_0 T} dk_0 dk$ This term is tricky.

$$\overset{\star}{=} \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{-ikX - ik_0 T} \left(\frac{1}{k_0 - k^2} + \frac{1}{2i} \delta(k_0 - k^2) \right) dk_0 dk$$

(One can check $(i\partial_T + \partial_X^2) \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{-ikX - ik_0 T} \cdot \delta(k_0 - k^2) dk_0 dk$

$= \int_{\mathbb{R}^2} \frac{1}{4\pi^2} e^{\cdots} (k_0 - k^2) \delta(k_0 - k^2) dk_0 dk = 0$)

Change variable $k_0 \to k_0 + k^2$

$$= \int_{\mathbb{R}} \frac{dk}{2\pi} e^{-ikX - ik^2 T} \boxed{\int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{-ik_0 T} \left(\frac{1}{k_0} + \frac{1}{2i} \delta(k_0) \right)}$$

$$\mathcal{F}\left(\frac{1}{k_0}\right)(T) + \frac{1}{2i} = \frac{1}{i} 1_{\{T>0\}}(T)$$

So $= 1_{\{T>0\}}(T) \frac{1}{i} \int_{\mathbb{R}} \frac{dk}{2\pi} e^{-ikX - ik^2 T} =$ what you want.

Fast way to show $\mathcal{F}(\frac{1}{x}) = \int_{\mathbb{R}} \frac{1}{x} e^{-itx} dx = \frac{1}{2i} \operatorname{sgn}(t)$:

Note $D_t \widehat{\frac{1}{x}} = \frac{1}{i} x \widehat{\frac{1}{x}} = \frac{1}{i} x \cdot \frac{1}{x} = \frac{1}{i} \Rightarrow D_t \widehat{\frac{1}{x}} = \frac{1}{i} \delta(t)$

$\Rightarrow \widehat{\frac{1}{x}} = \frac{1}{i} \frac{1}{D_t} \delta(t) = \frac{1}{i} \left(1_{\{t>0\}}(t) + C \right)$ since $\frac{1}{x}$ is an odd function.

We need $C = -\frac{1}{2}$. So $\widehat{\frac{1}{x}} = \frac{1}{2i} \operatorname{sgn}(t)$.