Sol 1: $\lim_{\epsilon\to 0} F_\epsilon = \delta$ ~~means~~ is defined as: for $\forall f \in C_c^\infty(\mathbb{R})$, $\lim_{\epsilon\to 0}\int_{\mathbb{R}} F_\epsilon f\,dx = \delta(f) = f(0)$ (*)

In order to show (*), first note: ① $\boxed{\int_{\mathbb{R}} F_\epsilon\,dx = 1}$ and ② $\forall\delta>0$ $\lim_{\epsilon\to 0}\int_{\mathbb{R}\setminus[-\delta,\delta]} F_\epsilon(x)\,dx = 0$

So for any small $\tau>0$, by continuity of f: $\exists\,\delta>0$ s.t. $\sup_{x\in[-\delta,\delta]}|f(x)-f(0)|<\tau$

18.305 Problem Set 5

So, $\int F_\epsilon f\,dx - f(0) \overset{\text{by ①}}{=} \int F_\epsilon(f(x)-f(0))\,dx = \int_{\mathbb{R}-[-\delta,\delta]} F_\epsilon(f(x)-f(0))\,dx + \int_{[-\delta,\delta]} F_\epsilon(f(x)-f(0))\,dx$

Mason Rogers

$\leq 2\sup|f(x)|\cdot\int_{\mathbb{R}-[-\delta,\delta]} F_\epsilon\,dx + \tau\int F_\epsilon\,dx$

By ②, we see $\lim_{\epsilon\to 0}\left|\int F_\epsilon\,dx - f(0)\right| < \tau$ since τ is arbitrary, we proved (*).

1

9/10

We wish to compute $F_\epsilon(x) = \int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\epsilon k^2}e^{ikx}dk$ and show that in the limit $\epsilon\to 0$, the integral converges to the Dirac delta. We will do so in three steps. First, we will evaluate the integral for $\epsilon>0$. Second, we will show that in the limit $\epsilon\to 0$, the integral vanishes for all $x\neq 0$ and is infinite for $x=0$, i.e. that $\lim_{\epsilon\to 0}F_\epsilon(x) = c\delta(x)$ for some c. Lastly, we will show by integrating with respect to x that $c=1$.

29/30

To evaluate the integral for $\epsilon>0$, we must complete the square of the exponent. Observe that

$$-\epsilon k^2 + ikx = -\epsilon\left[\left(k-\frac{ix}{2\epsilon}\right)^2 + \frac{x^2}{4\epsilon^2}\right].$$

We thus rearrange our original integral, revealing

$$F_\epsilon(x) = \frac{e^{-\frac{x^2}{4\epsilon}}}{2\pi}\int_{-\infty}^{\infty} e^{-\epsilon\left(k-\frac{ix}{2\epsilon}\right)^2}\,dk.$$

A substitution $k' = k - \frac{ix}{2\epsilon}$ yields a standard Gaussian integral, which evaluates to

$$F_\epsilon(x) = \frac{e^{-\frac{x^2}{4\epsilon}}}{2\pi}\cdot\sqrt{\frac{\pi}{\epsilon}} = \frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}}.$$

As our next step we take the limit when $\epsilon\to 0$. First, suppose $x\neq 0$. Defining $\mu = 1/\epsilon$ and manipulating the expression a bit yields

$$\lim_{\epsilon\to 0}\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} = \frac{1}{2\sqrt{\pi}}\lim_{\mu\to\infty}\frac{\sqrt{\mu}}{e^{\frac{x^2\mu}{4}}} = \frac{1}{x^2\sqrt{\pi}}\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}e^{\frac{x^2\mu}{4}}} = 0,$$

where in the last step we applied L'Hôpital's rule. Now suppose $x=0$, so $F_\epsilon(0) = 1/\sqrt{4\pi\epsilon}$. In this case,

$$\lim_{\epsilon\to 0}F_\epsilon(0) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{4\pi\epsilon}} = \infty.$$

From this, we conclude that $\lim_{\epsilon\to 0}F_\epsilon(x)$ vanishes everywhere except at $x=0$, where its value is infinite. Thus, $\lim_{\epsilon\to 0}F_\epsilon(x) = c\delta(x)$ for some constant c, and we must figure out what c is.

To show that $c = 1$, we will evaluate the integral $c = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} F_\epsilon(x)dx$. We have

$$c = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}}\,dx = \lim_{\epsilon\to 0}\frac{\sqrt{4\pi\epsilon}}{\sqrt{4\pi\epsilon}} = \lim_{\epsilon\to 0} 1 = 1.$$

We conclude that $\lim_{\epsilon\to 0} F_\epsilon(x) = \delta(x)$, as desired.

2

10/10

Observe that the potential $g|x|$ is symmetric about $x = 0$, so for any energy $E > 0$, there will be exactly two turning points. Thus, we will finish deriving the matching condition at the left turning point from class, which will give us an integral constraint on the energy eigenvalues. Evaluating the integral will allow us to calculate the energy eigenvalues explicitly.

Let x_0 and x_1 be the left and right turning points, respectively. Coming from the left of x_0, in the classically accessible region we have

$$\psi_{\text{wkb}}(x) = \frac{c\sin\left(\int_{x_0}^{x}\lambda\sqrt{E-V}\;dx' + \frac{\pi}{4}\right)}{\sqrt{\lambda}(E-V)^{1/4}}.$$

Coming from the right of x_1, we have another form for $\psi_{\text{wkb}}(x)$ in the classically accessible region, namely

$$\psi_{\text{wkb}}(x) = \frac{\sin\left(\int_{x}^{x_1}\lambda\sqrt{E-V}\;dx' + \frac{\pi}{4}\right)}{\sqrt{\lambda}(E-V)^{1/4}}.$$

Our matching condition is thus that for all $x \in [x_0, x_1]$,

$$c\sin\left(\int_{x_0}^{x}\lambda\sqrt{E-V}\;dx' + \frac{\pi}{4}\right) = \sin\left(\int_{x}^{x_1}\lambda\sqrt{E-V}\;dx' + \frac{\pi}{4}\right). \tag{1}$$

For brevity, let $I(x) = \int_{x_0}^{x}\lambda\sqrt{E-V}dx'$. Substituting $x = x_0$ in (1) reveals

$$c\sin\left(\frac{\pi}{4}\right) = \sin\left(I(x_1) + \frac{\pi}{4}\right).$$

Now setting $x = x_1$ in (1) and substituting the previous relation yields

$$c^2\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right),$$

or in other words $c = \pm 1$. Now we have reduced to solving

$$\pm\sin\left(I(x) + \frac{\pi}{4}\right) = \sin\left(I(x_1) - I(x) + \frac{\pi}{4}\right).$$

Applying the 'sine of sum' identity and observing that $\cos(\pi/4) = \sin(\pi/4)$ yields

$$\begin{aligned}\pm[\sin(I(x)) + \cos(I(x))) &= \sin(I(x_1) - I(x)) + \cos(I(x_1) - I(x))] \\ &= \sin(I(x_1))[\cos(I(x) + \sin(I(x))] \\ &\quad + \cos(I(x_1))[\cos(I(x)) - \sin(I(x))].\end{aligned}$$

This only holds for all x when $\sin(I(x_1)) = 1$ (and accordingly $\cos(I(x_1)) = 0$). Thus, we have $I(x_1) = (n + \frac{1}{2})\pi$ for $n \in \mathbb{N}$.

Now that we have derived the energy constraint in integral form, we can substitute the particular potential $V = g|x|$ and evaluate the integral. Observe that in this case, $x_1 = -x_0 = E_n/g$, where E_n is the n-th energy eigenvalue. We have

$$\begin{aligned} I(x_1) = (n + \frac{1}{2})\pi &= \int_{-E_n/g}^{E_n/g} \lambda\sqrt{E_n - g|x'|}\, dx' \\ &= 2\lambda \int_0^{E_n/g} \sqrt{E_n - g|x'|}\, dx' \\ &= \frac{4\lambda}{3g}(E_n)^{3/2} \\ \therefore E_n &= \left[\frac{3g\left(n + \frac{1}{2}\right)\pi}{4\lambda}\right]^{2/3}, \quad \lambda = \frac{m}{2\pi^2 h^2}. \end{aligned}$$

10/10

3

In this problem, we only have one turning point x_0, so our energy constraint will have to take a new form. Instead of solving a matching condition, we impose $\psi(0) = 0$ and obtain the restriction

$$\psi_{\text{wkb}}(0) = \sin\left(\int_0^{x_0} \lambda\sqrt{E + \frac{e^2}{r'}}\, dr' + \frac{\pi}{4}\right) = 0,$$

or after simplifying and setting $x_0 = -e^2/E_n$ to be the turning point corresponding to energy E_n,

$$\lambda e \int_0^{-e^2/E_n} \sqrt{\frac{E_n}{e^2} + \frac{1}{r'}}\, dr' = \left(n + \frac{3}{4}\right)\pi.$$

Making the substitution $u = \sqrt{-r'E_n/e^2}$ yields

$$\frac{\lambda e^2}{\sqrt{-E_n}} \int_0^1 \sqrt{1 - u^2}\, du = \frac{\pi\lambda e^2}{4\sqrt{-E_n}} = \left(n + \frac{3}{4}\right)\pi,$$

which rearranges to

$$E_n = -\frac{\lambda^2 e^4}{4\left(n + \frac{3}{4}\right)^2}.$$

The difference between this result and the actual result is that the WKB result has $(n + \frac{3}{4})^2$ in the denominator instead of $(n+1)^2$. This difference may be noticeable when the denominator is small, but it will become less significant for higher values of n.