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Problem 1

(a) The form of the WKB approximation for y is:

$$
\begin{aligned}
y(x) &\approx \frac{A}{\sqrt{x}} \exp\left(i \int_{x_0}^{x} x' dx'\right) + \frac{B}{\sqrt{x}} \exp\left(-i \int_{x_0}^{x} x' dx'\right) \\
&= \frac{A}{x} \exp\left(\frac{i}{2}(x^2 - x_0^2)\right) + \frac{B}{x} \exp\left(-\frac{i}{2}(x^2 - x_0^2)\right) \\
&= C\frac{1}{\sqrt{x}} \cos\left(\frac{1}{2}(x^2 - x_0^2)\right) + D\frac{1}{\sqrt{x}} \sin\left(\frac{1}{2}(x^2 - x_0^2)\right)
\end{aligned}
$$

We use the boundary conditions to determine the constants A and B.

$$
\begin{aligned}
y(x_0) &= C = \sqrt{x_0} \\
y'(x_0) &= D\sqrt{x_0} - C/x_0^{3/2} = 0 \quad \frac{C}{2x_0^{3/2}}
\end{aligned}
$$

This gives us that $C = \sqrt{x_0}$ and $D = \frac{x_0^{-3/2}}{2}$. Then,

$$
y(x) \approx \frac{\sqrt{x_0}}{\sqrt{x}} \cos\left(\frac{1}{3}(x^2 - x_0^2)\right) + \frac{1}{2x_0^{3/2}\sqrt{x}} \sin\left(\frac{1}{3}(x^2 - x_0^2)\right)
$$

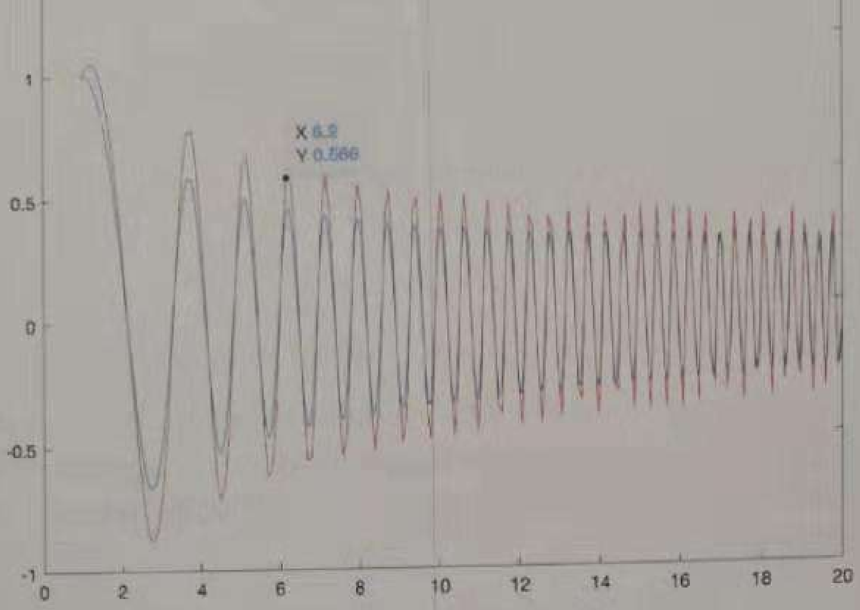
The validity condition for this approximation is:

$$
\left|\frac{d}{dx}\frac{1}{x}\right| = \left|\frac{-1}{x^2}\right| \ll 1
$$

This is equivalent to the condition:

$$
x \gg 1
$$

) Using Mathematica, we can find numerical approximations to the solution of t ferential equation. We compare this to the approximation by plotting both as their relative errors. (Mathematica code attached.)



igure 1: For $x_0 = 1$, the WKB approximation follows the exact solution decently well, but rly errors manifest in sustained incorrect amplitudes of oscillation.

(b) See Figure 1. For larger values of x_0, the condition $x^{-2} \leq x_0^{-2} \ll 1$ is more strongly satisfied, so the WKB approximation does a better job agreeing with a numerical solution to the ODE.

2. We begin with the problem

$$y^{(4)} + \lambda^4 U(x) y = 0.$$

Let $\phi = U^{1/4}$, and suppose $U(x) > 0$. Consider a solution of form

$$y = e^{\iota\lambda \int \phi(x')dx'} \left(v_0 + \frac{v_1}{\lambda} + \dots \right),$$

where ι is a solution to $\iota^4 = -1$. Substituting this form into our differential equation and dividing out the exponential factor, we obtain

$$v^{(4)} + \iota\lambda \left[\phi''' v + 4\phi'' v' + 6\phi' v'' + 4\phi v'''\right] + \iota^2\lambda^2 \left[3(\phi')^2 v + 4\phi\phi'' v + 12\phi\phi' v' + 6\phi^2 v''\right] \\ + \iota^3\lambda^3 \left[6\phi^2\phi' v + 4\phi^3 v'\right] = 0.$$

Since we assume $\lambda \gg 1$, we may solve for our first-order solution by collecting λ^3 terms. We obtain

$$6\phi^2\phi' v_0 + 4\phi^3 v_0' = 0,$$

which has solution $v_0 = -\frac{3}{2}\ln(\phi)$. Combining with the above, we have the first order solution

$$y \simeq e^{\iota\lambda \int \phi(x')dx'} (\phi^{-3/2}(x)).$$

Solution is $V_0 = \phi^{-3/2}$