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1 With the Schrodinger equation:

$$\left(i\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)\psi(x,t)=U(x,t)\psi(x,t)$$

Recall that the Green's Function for the Schrodinger equation is

$$G(x,t)=\frac{e^{-\frac{ix^2}{4t}}}{2\sqrt{-\pi i t}}$$

and that

$\psi_h = G(x,t) \underset{x}{*} f(x)$

$\left(i\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}\right)\psi(x,t)=\rho(x,t)$ is solved by $\psi(x,t)=\psi_h+G(x,t)*\int_0^t -i\rho(x,t')dt'$

If we say $U(x,t)=\epsilon V(x,t)$ with ϵ very small, and

$$\psi(x,t)=\psi^0(x,t)+\epsilon\psi^1(x,t)+\epsilon^2\psi^2(x,t)+...$$

Using this definition of ψ, we can substitute and say:

$$\left(\psi^0(x,t)+\epsilon\psi^1(x,t)+...\right)=\epsilon V(x,t)\left(\psi^0(x,t)+\epsilon\psi^1(x,t)+...\right)=$$

$$\psi_h-i\int_0^t G(x,t-t')*(-i\epsilon V\left(\psi^0(x,t)+\epsilon\psi^1(x,t)+...\right)$$

Equating terms of equal order ϵ, we get that $\psi^0=\psi_h$ and

$$\psi_{n+1}=\int_0^t dt' G(x,t-t')*V(x,t')\epsilon^n\psi^n(x,t')$$

2 Considering the heat equation, one must first solve the corresponding homogeneous equation:

$$\frac{\partial T_h(x,t)}{\partial t} - \frac{\partial^2 T_h(x,t)}{\partial x^2} = 0$$

Taking the Fourier transform of x to k, we get:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{T}_h(k,t) = 0 \Rightarrow \frac{\partial}{\partial x} = -ik$$

From the initial conditions, we know that as x goes to infinity in either direction, the integral converges, thus:

$$\left(\frac{\partial}{\partial t} + k^2\right) \tilde{T}_h(k,t) = 0$$

Assuming a solution is of the form e^{mk}, and solving the characteristic equation for $m = -k^2$, the following results:

$$\tilde{T}_h(k,t) = A(k)e^{k^2 t} \Rightarrow T_h(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} A(k) e^{-k^2 t}$$

Substituting in the initial condition $T(x,0) = f(x)$ yields:

$$T_h(x,0) = f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} A(k) \Rightarrow A(k) = \tilde{f}(k)$$

Thus the homogeneous solution is

$$T_h(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx' e^{-k^2 t} e^{ik(x-x')} f(x')$$

Switching the order of the integrals and calling the integration over k $G(x - x', t)$ we get:

$$G(x-x',t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2 t} e^{ik(x-x')}$$

Completing the square with the terms in the exponential:

$$-tk^2 + ik(x-x') = k^2 + k\frac{-i(x-x')}{t} = \left(k + \frac{-i(x-x')}{2t}\right)^2 - \frac{(x-x')^2}{4t^2}$$

Replacing back into the integral, and recalling that an integral of the Gaussian $e^{-a(x+b)^2}$ over $\mathbb{R}$ gives $\sqrt{\frac{\pi}{a}}$

$$G(x-x',t) = e^{-\frac{(x-x')^2}{4t}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{t\left(k+\frac{-i(x-x')}{2t}\right)^2} = e^{-\frac{(x-x')^2}{4t}} \frac{1}{2\pi}\sqrt{\frac{1}{\pi t}}$$

For the particular solution, after taking the Fourier transform of $\rho(x,t)$, recalling that $\frac{\partial}{\partial x} = -ik$, and saying $D = \frac{\partial}{\partial t}$ we get:

$$T_p(x,t) = \frac{1}{D+k^2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{\rho}(k,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{D+k^2} e^{k^2 t} e^{-k^2 t} e^{ikx} \tilde{\rho}(k,t)$$

Rearranging the exponential terms and $D + k^2$, then expanding $\tilde{\rho}(k,t)$:

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2 t} \frac{1}{D} e^{k^2 t} e^{ikx} \tilde{\rho}(k,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx' e^{-ikx'} e^{-k^2 t} \frac{1}{D} e^{k^2 t} e^{ikx} \tilde{\rho}(x,t)$$

Rearranging and integrating the $\frac{1}{D}$:

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-k^2 t} e^{ik(x-x')} \int_0^t dt' e^{k^2 t'} \rho(x',t')$$

Thus:

$$T_p(x,t) = \int_{-\infty}^{\infty} dx' \int_0^t dt' G(x-x',t-t') \rho(x',t') \text{ where } G(x-x',t-t') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} e^{-k^2(t-t')}$$

From the Green's function for $T_h(x,t)$, we get:

$$G(x-x',t-t') = e^{-\frac{(x-x')^2}{4(t-t')}} \sqrt{\frac{1}{4\pi(t-t')}}$$

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Problem Set 3

3 Expanding cosine into the exponential form yields:

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} cos(kt) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left(\frac{1}{2}\right) \left(e^{ikt} + e^{-ikt}\right)$$

Taking the limit as a goes to infinity, then evaluating the integral yields:

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{1}{2}\frac{dk}{2\pi} e^{ikx}(e^{ikt}+e^{-ikt}) = \left(\frac{1}{2i}\right)\left(\frac{1}{2\pi}\right) \lim_{a\to\infty} \left[\frac{1}{x+t} e^{ik(x+t)} + \frac{1}{x-t} e^{ik(x-t)}\right]_{k=-a}^{a}$$

Substituting in for k:

$$\left(\frac{1}{2i}\right)\left(\frac{1}{2\pi}\right) \lim_{a\to\infty} \left[\frac{1}{x+t}\left(e^{ia(x+t)} + e^{-ia(x+t)}\right) + \frac{1}{x-t}\left(e^{ia(x-t)} + e^{-ia(x-t)}\right)\right] =$$

Simplifying exponential and recalling that a definition of the Dirac Delta function is $\delta(x) = \lim_{\epsilon\to 0} \frac{sin(x/\epsilon)}{\pi x}$

$$\frac{1}{2\pi} \lim_{a\to\infty} \frac{sin(a(x+t))}{x+t} + \frac{sin(a(x-t))}{x-t} = \frac{1}{2}\left(\delta(x+t) + \delta(x-t)\right).$$

Physical explanation:

The waveform can be decomposed into a left traveling wave form & right traveling waveform originating at t=0 tx