18.305 Problem Set 2

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We aim to solve

$$\Delta u = 0; \quad u(x,y) \to 0 \text{ as } x \to \pm\infty, u(x,0) = f(x), u(x,a) = 0.$$

We jump in by taking the Fourier transform of both sides, allowing us to replace ∂_x with ik and yielding

$$(\partial_y^2 - k^2)\hat{u} = 0; \quad \hat{u}(k,y) \to 0 \text{ as } k \to \pm\infty, \hat{u}(k,0) = \hat{f}(k), \hat{u}(k,a) = 0.$$

The transformed PDE has solution $\hat{u} = b(k)e^{-ky} + c(k)e^{ky}$, where b and c will be determined by the boundary conditions. Indeed, substituting our conditions for $\hat{u}(k,0)$ and $\hat{u}(k,a)$ gives the system of equations

$$b(k) + c(k) = \hat{f}(k)$$
$$b(k)e^{-ka} + c(k)e^{ka} = 0.$$

Solving and rearranging yields $b(k) = (1-e^{-2ka})^{-1}\hat{f}(k)$ and $c(k) = (1-e^{2ka})^{-1}\hat{f}(k)$. Substituting into our expression for $\hat{u}(k,y)$ and simplifying gives

$$\hat{u}(k,y) = \frac{e^{-ky}\hat{f}(k)}{1-e^{-2ka}} + \frac{e^{ky}\hat{f}(k)}{1-e^{2ka}} = \frac{\sinh(k(a-y))}{\sinh(ka)}\hat{f}(k).$$

Should
when k
eqn is
i.e

no derivative

As a technical point, we can define $\hat{u}(0,y)$ to be $(a-y)\hat{f}'(0)/a$, the limit of $\sinh(k(a-y))\hat{f}(k)/\sinh(ka)$ as $k \to 0$. Since $0 \le y \le a$, we have that $a > a - y$ and accordingly that $\hat{u}(k,y)$ tends to 0 as $k \to \pm\infty$ sufficiently quickly for its inverse Fourier transform to converge. The other boundary conditions are satisfied by construction, so all we are left to do is evaluate the inverse Fourier transform of $\hat{u}$. We note that $u(x,y) = G(x,y) * f(x)$, where '$*$' denotes convolution and

$$G(x,y) = \mathcal{F}^{-1}\left(\frac{\sinh(k(a-y))}{\sinh(ka)}\right) = \frac{1}{2a} \cdot \frac{\sin(\pi(a-y)/a)}{\cos(\pi(a-y)/a) + \cosh(\pi x/a)}.$$

We aim to solve

$$(i\partial_t + \partial_x^2)\psi = \rho(x,t); \quad \psi(x,0) = f(x).$$

We will do this by writing $\psi = \psi_h + \psi_p$, where ψ_h satisfies the corresponding homogeneous initial value problem

$$(i\partial_t + \partial_x^2)\psi_h = 0; \quad \psi_h(x,0) = f(x)$$

and ψ_p denotes the inhomogeneous solution with trivial initial conditions, i.e.

$$(i\partial_t + \partial_x^2)\psi_p = \rho(x,t); \quad \psi_p(x,0) = 0.$$

To solve for ψ_h, we take the Fourier transform in x of both sides and divide by i, leading to the initial value problem

$$\partial_t\hat{\psi_h} + ik^2\hat{\psi_h} = 0; \quad \hat{\psi_h}(k,0) = \hat{f}(k).$$

We see that $\hat{\psi_h} = \hat{f}(k)e^{-ik^2t}$, and accordingly $\psi_h(x,t) = f(x) *_x G(x,t)$, where

$$G(x,t) = \mathcal{F}^{-1}\left(e^{-ik^2t}\right) = \frac{e^{-\frac{ix^2}{4t}}}{2\sqrt{-\pi i t}}.$$

We use the notation '$*_x$' to denote convolution in the first argument integrating over $(-\infty,\infty)$, to be contrasted with '$*_t$' denoting convolution in the second argument integrating over $[0,t]$.

Now we seek ψ_p as speficied above. We begin again with the Fourier transform and dividing by i, giving

$$\partial_t\hat{\psi_p} + ik^2\hat{\psi_p} = -i\hat{\rho}(k,t); \quad \hat{\psi_p}(k,0) = 0.$$

Solving, we see

$$\begin{aligned}
(D + ik^2)\hat{\psi_p} &= -i\hat{\rho} \\
\hat{\psi_p} &= \frac{1}{D+ik^2}e^{-ik^2t}e^{ik^2t}(-i\hat{\rho}) \\
\hat{\psi_p} &= e^{-ik^2t}\int_0^t e^{ik^2t'}(-i\hat{\rho}(k,t'))\,dt' \\
\hat{\psi_p} &= \int_0^t \hat{G}(k,t-t')(-i\hat{\rho}(k,t'))\,dt' \\
\hat{\psi_p} &= \hat{G} *_t (-i\hat{\rho}) \\
\psi_p &= G(x,t) *_{x,t} (-i\rho(x,t)).
\end{aligned}$$

Observe that our initial conditions for ψ_p are satisfied by the choice of our limits of integration in t'. Thus, we combine to obtain

$$\psi = \psi_h + \psi_p = G(x,t) *_x f(x) + G(x,t) *_{x,t} (-i\rho(x,t)).$$