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Problem 1

In the region $0 < x < 1$, we want to solve

$$\epsilon y'' - y' + (1 + x^3)y = 0 \qquad \epsilon \ll 1$$

for the boundary conditions: $y(0) = 1$ and $y(1) = 3$. In this case, $a(x) = -1 < 0$ over the entire region, so y_r is rapidly increasing:

$$\begin{aligned} y_r &= \frac{c_r}{a(x)} \exp \int \left(-\frac{a}{\epsilon} + \frac{b}{a} \right) dx \\ &= -c_r \exp \int \left(\frac{1}{\epsilon} - (1 + x^3) \right) dx \end{aligned}$$

The main contribution to the integral from y_r then comes only from the upper endpoint, at $x = 1$. So, there is a boundary layer at $x = 1$ with width of order ϵ. Using the boundary conditions, we find:

$$\begin{aligned} y(0) &= y_s(0) + y_r(0) \approx y_s(0) = 1 \\ y(1) &= y_s(1) + y_r(1) = 3 \end{aligned}$$

Outside the boundary layer, the solution is just $y(x) \approx y_s(x)$,

$$y_{out} = y_s(x) = \exp\left(\int_0^x (1 + x^3) dx \right) = \exp\left(x + \frac{1}{4} x^4 \right)$$

Inside the small boundary layer, we can approximate $y_s(x) \approx y_s(1)$, so

$$\begin{aligned} y_{in} &= y_s(1) + y_r(x) = y_s(1) + (3 - y_s(1)) \exp\left(\frac{x-1}{\epsilon} \right) \\ &= e^{5/4} + (3 - e^{5/4}) \exp\left(\frac{x-1}{\epsilon} \right) \end{aligned}$$

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Problem 2

In the region $0 < x < 1$, we want to solve

$$\epsilon y'' + 2y' + (1 + x^3)y = 0 \qquad \epsilon \ll 1$$

for the boundary conditions: $y(0) = 1$ and $y(1) = 0$. In this case, $a(x) = 2 > 0$ over the entire region, so y_r is rapidly decreasing.

$$y_r = \frac{c_r}{a(x)} \exp \int \left(-\frac{a}{\epsilon} + \frac{b}{a}\right) dx$$
$$= \frac{c_r}{2} \exp \int \left(-\frac{2}{\epsilon} + \frac{1+x^3}{2}\right) dx$$

The main contribution to the integral from y_r then comes only from the lower endpoint, at $x = 0$. So, there is a boundary layer at $x = 0$ with width of order $\cancel{\epsilon}$. Using boundary conditions,

1 ← Since $y_s = 0$

(not asked in the question, so I can give the full points)

$$y(0) = y_s(0) + y_r(0) = 1$$
$$y(1) = y_s(1) + y_r(1) \approx y_s(1) = 0$$

We have that y_{out} satisfies the differential equation:

$$2y'_{out} + (1 + x^3)y_{out} = 0$$

So the solution is

$$y_{out} = \exp\left(-\int_0^x \frac{1+x^3}{2} dx\right) = c \exp\left(-\frac{1}{2}x - \frac{1}{8}x^4\right)$$

Since this holds for $x > 0$, we have

$$y_{out}(1) = c \exp\left(-\frac{5}{8}\right) = 0 \quad \to \quad c = 0$$

Inside the boundary layer,

$$y_{in}(x) = c_2 \exp\left(-\frac{2x}{\epsilon}\right)$$

which must satisfy the boundary condition

$$y_{in}(0) = 1 \quad \to \quad c_2 = 1$$

Then,

$$y_{in}(x) = \exp\left(-\frac{2x}{\epsilon}\right)$$ ✓

Both $y_{out}(x) = 0$ or $y_{out}(x) = \exp(-\frac{2x}{\epsilon})$ are OK.