

18.303 Fall 2009, Problem Set 5 Solutions

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Let $k = ax$, then

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dk \frac{1}{(a - ik)^2(a + ik)^2} \\ &= \frac{1}{a^3} \int_{-\infty}^{\infty} dx \frac{1}{(x - i)^2(x + i)^2} \end{aligned}$$

Rewriting the integrand by partial fraction expansion,

$$\frac{1}{(x - i)^2(x + i)^2} = \frac{i}{4(x + i)} - \frac{1}{4(x + i)^2} - \frac{i}{4(x - i)} - \frac{1}{4(x - i)^2}$$

We now evaluate the integral by contour integration. The real axis is the contour from $-R$ to R along the real axis as R goes to infinity. We close the contour by adding the counterclockwise semicircle arc from R to $-R$ in the upper-half plane. We then have a closed contour and the integral over the semicircle arc vanishes as $R \rightarrow \infty$. Thus we have,

$$I = \frac{1}{a^3} \oint dz \left[\frac{i}{4(z + i)} - \frac{1}{4(z + i)^2} - \frac{i}{4(z - i)} - \frac{1}{4(z - i)^2} \right]$$

The closed contour encloses the pole $z_0 = i$. $f(z) = q/(z - z_0)^2 + r/(z - z_0) + \text{non-singular terms}$ near $z = i$, where $q = -1/4, r = -i/4$. Using the hint given in the problem statement,

$$\begin{aligned} I &= \frac{1}{a^3} (2\pi i r) \\ &= \frac{1}{a^3} [2\pi i (-i/4)] \\ &= \frac{\pi}{2a^3} \end{aligned}$$

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(a)

$$\begin{aligned}
G(x, y, t) &= \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \\
&\int_{-\infty}^{\infty} dy G(x, y, t_2 - t_1) G(y, z, t_1 - t_0) \\
&= \frac{1}{4\pi k \sqrt{(t_2 - t_1)(t_1 - t_0)}} \int_{-\infty}^{\infty} dy \exp \left[-\frac{(x-y)^2}{4k(t_2 - t_1)} - \frac{(y-z)^2}{4k(t_1 - t_0)} \right] \\
&= \frac{1}{4\pi k \sqrt{(t_2 - t_1)(t_1 - t_0)}} \int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_1 - t_0)(x^2 - 2xy + y^2) + (t_2 - t_1)(y^2 - 2zy + z^2)}{4k(t_2 - t_1)(t_1 - t_0)} \right] \\
&= \frac{1}{4\pi k \sqrt{(t_2 - t_1)(t_1 - t_0)}} \int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_2 - t_0)y^2 - 2[(t_1 - t_0)x + (t_2 - t_1)z]y + (t_1 - t_0)x^2 + (t_2 - t_1)z^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \\
&= \frac{1}{4\pi k \sqrt{(t_2 - t_1)(t_1 - t_0)}} \exp \left[-\frac{(t_1 - t_0)x^2 + (t_2 - t_1)z^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_2 - t_0)y^2 - 2[(t_1 - t_0)x + (t_2 - t_1)z]y}{4k(t_2 - t_1)(t_1 - t_0)} \right] \tag{1}
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_2 - t_0)y^2 - 2[(t_1 - t_0)x + (t_2 - t_1)z]y}{4k(t_2 - t_1)(t_1 - t_0)} \right] \\
&= \int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_2 - t_0)(y - A)^2 - (t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} \right], \quad \text{where } A = \frac{(t_1 - t_0)x + (t_2 - t_1)z}{t_2 - t_0} \\
&= \exp \left[\frac{(t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \int_{-\infty}^{\infty} dy \exp \left[-\frac{(t_2 - t_0)(y - A)^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \\
&= \exp \left[\frac{(t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \int_{-\infty}^{\infty} dz e^{-z^2} \left(2\sqrt{\frac{k(t_2 - t_1)(t_1 - t_0)}{t_2 - t_0}} \right), \quad \text{letting } z = \frac{\sqrt{t_2 - t_0}(y - A)}{2\sqrt{k(t_2 - t_1)(t_1 - t_0)}} \\
&= \exp \left[\frac{(t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} \right] \left(2\sqrt{\frac{\pi k(t_2 - t_1)(t_1 - t_0)}{t_2 - t_0}} \right) \tag{2}
\end{aligned}$$

Substituting (2) into (1),

$$\begin{aligned}
&\int_{-\infty}^{\infty} dy G(x, y, t_2 - t_1) G(y, z, t_1 - t_0) \\
&= \frac{1}{\sqrt{4\pi k(t_2 - t_0)}} \exp \left[-\frac{(t_1 - t_0)x^2 + (t_2 - t_1)z^2 - (t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} \right]
\end{aligned}$$

Simplifying the term of the exponent (details omitted), we will get

$$-\frac{(t_1 - t_0)x^2 + (t_2 - t_1)z^2 - (t_2 - t_0)A^2}{4k(t_2 - t_1)(t_1 - t_0)} = -\frac{(x - z)^2}{4k(t_2 - t_0)}$$

Thus

$$\int_{-\infty}^{\infty} dy G(x, y, t_2 - t_1) G(y, z, t_1 - t_0) = \frac{1}{\sqrt{4\pi k(t_2 - t_0)}} e^{-\frac{(x-z)^2}{4k(t_2 - t_0)}} = G(x, z, t_2 - t_0)$$

(b) The spectral form of $G(x, y, t)$ for heat flow on a finite interval is

$$G(x, y, t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \frac{\phi_n(x)\phi_n(y)}{N_n^2}$$

We show that the Green's function satisfies the “time composition” property:

$$\begin{aligned} & \int_{-\infty}^{\infty} dy G(x, y, t_2 - t_1) G(y, z, t_1 - t_0) \\ &= \int_{-\infty}^{\infty} dy \left(\sum_{n=1}^{\infty} e^{-\lambda_n(t_2-t_1)} \frac{\phi_n(x)\phi_n(y)}{N_n^2} \right) \left(\sum_{m=1}^{\infty} e^{-\lambda_m(t_1-t_0)} \frac{\phi_m(y)\phi_m(z)}{N_m^2} \right) \end{aligned} \quad (3)$$

Since the eigenfunctions ϕ_n satisfies the orthogonality conditions:

$$\int_{-\infty}^{\infty} dy \phi_n(y)\phi_m(y) = N_n^2 \delta_{nm},$$

the only nonzero terms in the products of the two series in equation (3) are the ones corresponding to $n = m$. This gives us

$$\begin{aligned} & \int_{-\infty}^{\infty} dy G(x, y, t_2 - t_1) G(y, z, t_1 - t_0) \\ &= \int_{-\infty}^{\infty} dy \sum_{n=1}^{\infty} \left(e^{-\lambda_n(t_2-t_1)} \frac{\phi_n(x)\phi_n(y)}{N_n^2} \right) \left(e^{-\lambda_n(t_1-t_0)} \frac{\phi_n(y)\phi_n(z)}{N_n^2} \right) \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n(t_2-t_0)} \frac{\phi_n(x)\phi_n(z)}{N_n^4} \int_{-\infty}^{\infty} dy \phi_n(y)\phi_n(y) \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n(t_2-t_0)} \frac{\phi_n(x)\phi_n(z)}{N_n^2} \\ &= G(x, z, t_2 - t_0) \end{aligned}$$

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The one-dimensional Schrödinger equation:

$$\begin{aligned} i \frac{\partial \Psi}{\partial t} &= -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + V(x) \Psi \\ \text{where } V(x) &= \begin{cases} 0, & -a \leq x \leq a \\ V_0, & |x| > a \end{cases} \end{aligned}$$

We consider the energy, i.e. the eigenvalue of the ODE which satisfies $E > V_0$. We require $|\Psi|$ remains bounded as $x \rightarrow \infty$. Applying separation of variables:

$$\begin{aligned} \Psi(x, t) &= \phi(x)T(t) \\ i\phi\dot{T} &= -\frac{1}{2}\phi''T + V(x)\phi T \\ i\frac{\dot{T}}{T} &= -\frac{1}{2}\frac{\phi''}{\phi} + V(x) = E \end{aligned}$$

We then have

$$\begin{aligned}\frac{\dot{T}}{T} &= -iE \\ T(t) &= e^{-iEt} \\ -\frac{1}{2}\phi'' + V(x)\phi &= E\phi \\ \phi'' + 2(E - V)\phi &= 0\end{aligned}\tag{4}$$

We only consider solutions for which the energies satisfy $E > V_0$. For $|x| \leq a$, $V(x) = 0$, thus we have $\phi = A \cos(k_{in}x) + B \sin(k_{in}x)$, where $k_{in} = \sqrt{2E}$. For $|x| > a$, we have $\phi = C \exp(-ik_{out}x) + D \exp(ik_{out}x)$, where $k_{out} = \sqrt{2(E - V_0)}$. Note that ϕ remains bounded as $|x| \rightarrow \infty$. A symmetric potential $V(x)$ allows the solution to be classified into two symmetric classes: either even or odd parity. For simplicity, we set $C = 1$.

Consider ϕ even, then $\phi = A \cos(k_{in}x)$, for $|x| \leq a$. For $x > a$, $\phi = \exp(-ik_{out}x) + D \exp(ik_{out}x)$. Requiring the solution to be continuous at $x = a$:

$$\begin{aligned}A \cos(k_{in}a) &= e^{-ik_{out}a} + D e^{ik_{out}a} \\ -k_{in}A \sin(k_{in}a) &= -ik_{out}e^{-ik_{out}a} + ik_{out}D e^{ik_{out}a}.\end{aligned}$$

Therefore,

$$\begin{aligned}A &= \frac{2ik_{out}e^{-ik_{out}a}}{ik_{out} \cos(k_{in}a) + k_{in} \sin(k_{in}a)} \\ D &= e^{-2ik_{out}a} \cdot \frac{ik_{out} - k_{in}}{ik_{out} + k_{in}}.\end{aligned}$$

Thus for even parity,

$$\phi_{even}(x) = \begin{cases} A \cos(k_{in}x), & |x| \leq a \\ e^{-ik_{out}x} + D e^{ik_{out}x}, & x > a \\ e^{ik_{out}x} + D e^{-ik_{out}x}, & x < -a. \end{cases}\tag{5}$$

Consider ϕ odd, then $\phi = B \sin(k_{in}x)$, for $|x| \leq a$. For $x > a$, $\phi = \exp(-ik_{out}x) + D \exp(ik_{out}x)$. Requiring the solution to be continuous at $x = a$:

$$\begin{aligned}B \sin(k_{in}a) &= e^{-ik_{out}a} + D e^{ik_{out}a} \\ k_{in}B \cos(k_{in}a) &= -ik_{out}e^{-ik_{out}a} + ik_{out}D e^{ik_{out}a}.\end{aligned}$$

Therefore,

$$\begin{aligned}B &= \frac{2ik_{out}e^{-ik_{out}a}}{ik_{out} \sin(k_{in}a) - k_{in} \cos(k_{in}a)} \\ D &= e^{-2ik_{out}a} \cdot \frac{ik_{out} + k_{in}}{ik_{out} - k_{in}}.\end{aligned}$$

Thus for odd parity,

$$\phi_{odd}(x) = \begin{cases} B \sin(k_{in}x), & |x| \leq a \\ e^{-ik_{out}x} + D e^{ik_{out}x}, & x > a \\ -e^{ik_{out}x} - D e^{-ik_{out}x}, & x < -a. \end{cases}\tag{6}$$

Combining equations (4), (5) and (6), we get the solution of the Schrödinger equation:

$$\boxed{\Psi(x, t) = e^{-iEt}(\phi_{\text{even}}(x) + \phi_{\text{odd}}(x))} \quad (7)$$

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$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi \\ \frac{\partial u}{\partial x}(0, t) &= A(t), & \frac{\partial u}{\partial x}(\pi, t) = B(t) \end{aligned}$$

First we consider the heat equation with homogenous Neumann boundary conditions. It is easy to show that the eigenvalues and eigenfunctions are

$$\begin{aligned} \lambda_n &= n^2 \\ \phi_n(x) &= \cos nx \end{aligned}$$

The Green's function for the homogeneous system is thus

$$\boxed{G_N(x, y, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos(nx) \cos(ny)} \quad (8)$$

We now solve the original inhomogenous system in terms of the Green's function. We present a similar derivation to the Dirichlet case (notes pp 99-100).

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} b_n(t) \phi_n(x) \\ b_n(t) &= \frac{1}{N_n^2} \int_0^{\pi} dx \phi_n(x) u(x, t) \\ \dot{b}_n(t) &= \frac{1}{N_n^2} \int_0^{\pi} dx \phi_n(x) \partial_t u(x, t) \\ &= \frac{1}{N_n^2} \int_0^{\pi} dx \phi_n(x) \partial_x^2 u(x, t) \end{aligned}$$

Using Green's formula for Sturm-Liouville problems,

$$\int_a^b dx [v(x) Lu(x) - Lv(x) u(x)] = p(x) (v \partial_x u - u \partial_x v) \Big|_a^b$$

In this case, $v = \phi_n$, $L = \frac{\partial^2}{\partial x^2}$ and $L\phi_n = -\lambda_n \phi_n$, $p(x) = 1$. Then

$$\begin{aligned} \dot{b}_n(t) &= \frac{1}{N_n^2} \left\{ -\lambda_n \int_0^{\pi} dx \phi_n(x) u(x, t) + [\phi_n(x) \partial_x u(x, t) - u(x, t) \partial_x \phi_n(x)]_0^{\pi} \right\} \\ &= -\lambda_n b_n(t) + \frac{1}{N_n^2} [B(t) \phi_n(\pi) - A(t) \phi_n(0)] \end{aligned}$$

Let $f_n(t) \equiv \frac{1}{N_n^2}[B(t)\phi_n(\pi) - A(t)\phi_n(0)]$,

$$\begin{aligned}
 \dot{b}_n(t) &= -\lambda_n b_n(t) + f_n(t) \\
 b_n(t) &= e^{-\lambda_n t} \int_0^t dt' e^{\lambda_n t'} f_n(t') \\
 u(x, t) &= \sum_{n=1}^{\infty} b_n(t) \phi_n(x) \\
 &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_0^t dt' e^{\lambda_n t'} f_n(t') \phi_n(x) \\
 &= \int_0^t dt' \sum_{n=1}^{\infty} e^{-\lambda_n(t-t')} f_n(t') \phi_n(x) \\
 &= \int_0^t dt' \sum_{n=1}^{\infty} \frac{e^{-\lambda_n(t-t')}}{N_n^2} \phi_n(x) [B(t')\phi_n(\pi) - A(t')\phi_n(0)] \\
 &= \int_0^t dt' [G_N(x, \pi, t-t')B(t') - G_N(x, 0, t-t')A(t')]
 \end{aligned}$$

$$\boxed{u(x, t) = \int_0^t dt' [G_N(x, \pi, t-t')B(t') - G_N(x, 0, t-t')A(t')]} \quad (9)$$

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$$F(\mathbf{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x})$$

Using spherical coordinates and choosing the z-axis in the same direction as the wave-vector $\mathbf{k}$ of the Fourier transform,

$$\mathbf{k} \cdot \mathbf{x} = kr \cos \phi,$$

where $k = |\mathbf{k}|$.

$$\begin{aligned}
 F(\mathbf{k}) &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-ikr \cos \phi} r^2 \sin \phi dr d\phi d\theta \\
 &= 2\pi \int_0^{\pi} \sin \phi d\phi \int_0^{\infty} r^2 e^{-(ik \cos \phi + 1/a)r} dr \\
 &= 2\pi \int_0^{\pi} d\phi \sin \phi \frac{2}{(ik \cos \phi + 1/a)^3} \quad (\text{Integration by parts twice}) \\
 &= 2\pi \left[\frac{1}{ik(ik \cos \phi + 1/a)^2} \right]_0^{\pi} \\
 &= \frac{2\pi}{ik} \left[\frac{1}{(1/a - ik)^2} - \frac{1}{(1/a + ik)^2} \right] \\
 &= \frac{2\pi}{ik} \frac{4ik/a}{(1/a^2 + k^2)^2} \\
 &= \frac{8\pi a^3}{(1 + a^2 k^2)^2}
 \end{aligned}$$

Thus the three-dimensional Fourier Transform is

$$\boxed{F(\mathbf{k}) = \frac{8\pi a^3}{(1 + a^2 k^2)^2}, \quad \text{where } k = |\mathbf{k}|} \quad (10)$$

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Consider the 2D Laplace equation

$$\begin{aligned} \nabla^2 u(\vec{x}) &= 0 \quad \text{in } \mathcal{R} \\ \hat{n} \cdot \nabla u(\vec{x})|_{\partial\mathcal{R}} &= -\Phi(\vec{x}) \end{aligned}$$

Using Green's second identity,

$$\int_{\mathcal{R}} d^2x [u(\vec{x}) \nabla^2 v(\vec{x}) - v(\vec{x}) \nabla^2 u(\vec{x})] = \int_{\partial\mathcal{R}} dx [u(\vec{x}) \hat{n} \cdot \nabla v(\vec{x}) - v(\vec{x}) \hat{n} \cdot \nabla u(\vec{x})]$$

Choose $u(\vec{x})$ to be the solution to our PDE and $v(\vec{x}) = G_N(\vec{x}, \vec{y})$,

$$\begin{aligned} \int_{\mathcal{R}} d^2x u(\vec{x}) \delta(\vec{x} - \vec{y}) &= - \int_{\partial\mathcal{R}} dx G_N(\vec{x}, \vec{y}) \hat{n} \cdot \nabla u(\vec{x}) \\ u(\vec{y}) &= - \int_{\partial\mathcal{R}} dx [\hat{n} \cdot \nabla u(\vec{x})] G_N(\vec{x}, \vec{y}) \\ &= \int_{\partial\mathcal{R}} dx \Phi(\vec{x}) G_N(\vec{x}, \vec{y}) \\ u(\vec{x}) &= \int_{\partial\mathcal{R}} dy \Phi(\vec{y}) G_N(\vec{y}, \vec{x}) \end{aligned}$$

Using symmetry of the Green's function,

$$\boxed{u(\vec{x}) = \int_{\partial\mathcal{R}} dy \Phi(\vec{y}) G_N(\vec{x}, \vec{y})} \quad (11)$$