

18.303 Fall 2009, Problem Set 3 Solutions

11 Riemann-Lebesgue Lemma

(a) If $f(x)$ and $f'(x)$ are continuous on $[a, b]$, then we can apply integration on the integral

$$\int_a^b dx \cos(kx) f(x) = \left[\frac{f(x) \sin(kx)}{k} \right]_a^b - \int_a^b \frac{f'(x) \sin(kx)}{k} dx$$
$$\left| \int_a^b dx \cos(kx) f(x) \right| \leq \left| \frac{f(b) \sin(kb) - f(a) \sin(ka)}{k} \right| + \frac{1}{k} \int_a^b |f'(x) \sin(kx)| dx,$$

which vanishes as $k \rightarrow \infty$ at least as fast as $1/k$. Similar argument can be applied to the integral $\int_a^b dx \sin(kx) f(x)$. $\square$

(b) In addition, if $f(a) = f(b) = 0$, and $f''(x)$ is continuous, we have

$$\begin{aligned} \int_a^b dx \cos(kx) f(x) &= \left[\frac{f(x) \sin(kx)}{k} \right]_a^b - \int_a^b \frac{f'(x) \sin(kx)}{k} dx \\ &= - \int_a^b \frac{f'(x) \sin(kx)}{k} dx \\ &= \left[\frac{f'(x) \cos(kx)}{k^2} \right]_a^b - \int_a^b \frac{f''(x) \cos(kx)}{k^2} dx \end{aligned}$$

Then

$$\left| \int_a^b dx \cos(kx) f(x) \right| \leq \left| \frac{f'(b) \cos(kb) - f'(a) \cos(ka)}{k^2} \right| + \frac{1}{k^2} \int_a^b |f''(x) \cos(kx)| dx,$$

which vanishes as $k \rightarrow \infty$ at least as fast as $1/k^2$. Similar argument can be applied to the integral $\int_a^b dx \sin(kx) f(x)$.

So the conditions are: $f(a) = f(b) = 0$ and $f''(x)$ is continuous on $[a, b]$.

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At equilibrium, $\partial_t u = 0$. So we have $u = u(x)$.

(a)

$$\begin{aligned}
\frac{d^2u}{dx^2} + 1 &= 0 \\
\frac{du}{dx} &= -x + C_1 \\
u &= -\frac{x^2}{2} + C_1x + C_2 \\
\frac{du}{dx}(0) &= 1 = C_1 \\
\frac{du}{dx}(L) &= -L + 1 = \beta
\end{aligned}$$

So if $\beta = 1 - L$, we have the equilibrium solution

$$\boxed{u = -\frac{x^2}{2} + x + C_2} \tag{1}$$

Physically, heat flux out (1 at left endpoint) equals heat flux in ($1 - L$ at the right end point) and heat produced by the source (L).

(b)

$$\begin{aligned}
\frac{d^2u}{dx^2} &= 0 \\
\frac{du}{dx} &= C_1 \\
u &= C_1x + C_2 \\
\frac{du}{dx}(0) &= 1 = C_1 \\
\frac{du}{dx}(L) &= 1 = \beta
\end{aligned}$$

So if $\beta = 1$, we have the equilibrium solution

$$\boxed{u = x + C_2} \tag{2}$$

In this case, heat flow in equals heat flow out.

(c)

$$\begin{aligned}
\frac{d^2u}{dx^2} &= -x + \beta \\
\frac{du}{dx} &= -\frac{x^2}{2} + \beta x + C_1 \\
u &= -\frac{x^3}{6} + \beta \frac{x^2}{2} + C_1x + C_2 \\
\frac{du}{dx}(0) &= C_1 = 0 \\
\frac{du}{dx}(L) &= -\frac{L^2}{2} + \beta L = 0
\end{aligned}$$

So if $\beta = \frac{L}{2}$, we have the equilibrium solution

$$\boxed{u = -\frac{x^3}{6} + \frac{Lx^2}{4} + C_2} \quad (3)$$

In this case, the influx and outflux of heat are both zero. The heat produced by the sources must add up to zero (we have two source terms, one positive and one negative).

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Suppose u_1 and u_2 are solutions to the equation. Let $U = u_1 - u_2$, then we have

$$\begin{aligned} \frac{\partial}{\partial t} U(x, t) &= \frac{\partial^2}{\partial x^2} U(x, t) \\ \frac{\partial U}{\partial x}(0, t) &= H(t)U(0, t) \\ U(L, t) &= 0 \\ U(x, 0) &= 0 \end{aligned}$$

Consider

$$E(t) = \int_0^L \frac{1}{2} U^2 dx \geq 0$$

Then by the initial condition,

$$E(0) = \int_0^L \frac{1}{2} (U(x, 0))^2 dx = 0$$

We also have

$$\begin{aligned} \frac{dE}{dt} &= \int_0^L U \frac{\partial U}{\partial t} dx \\ &= \int_0^L U \frac{\partial^2 U}{\partial x^2} dx \\ &= \left[U \frac{\partial U}{\partial x} \right]_0^L - \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \\ &= -U(0, t) \frac{\partial U}{\partial x}(0, t) - \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \\ &= -H(t)(U(0, t))^2 - \int_0^L \left(\frac{\partial U}{\partial x} \right)^2 dx \\ &\leq 0 \end{aligned}$$

So we must have $E(t) = 0$ for all t . Thus $U = 0$, which means $u_1 = u_2$. □

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$$\begin{aligned} \frac{\partial}{\partial t} \int_0^L u(x, t) dx &= \int_0^L \frac{\partial}{\partial t} u(x, t) dx = \int_0^L \frac{\partial^2}{\partial x^2} u(x, t) dx = \Phi_0 - \Phi_L \\ \int_0^L u(x, t) dx &= (\Phi_0 - \Phi_L)t + c \end{aligned}$$

So $u(x, t)$ has the form

$$u(x, t) = \frac{\Phi_0 - \Phi_L}{L}t + u_s(x) + v(x, t),$$

where $v(x, t)$ vanishes as $t \rightarrow \infty$.

$$\frac{\partial u}{\partial t} = \frac{\Phi_0 - \Phi_L}{L} + \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u_s}{dx^2} + \frac{\partial^2 v}{\partial x^2}$$

At large t , we have

$$\begin{aligned} \frac{\Phi_0 - \Phi_L}{L} &= \frac{d^2 u_s}{dx^2} \\ \frac{du_s}{dx} &= \frac{\Phi_0 - \Phi_L}{L}x + a \\ u_s &= \frac{\Phi_0 - \Phi_L}{2L}x^2 + ax + b \\ \frac{du_s}{dx}(0) &= a = -\Phi_0 \\ \frac{du_s}{dx}(L) &= (\Phi_0 - \Phi_L) + a = -\Phi_L \end{aligned}$$

Thus at large t ,

$$u(x, t) \rightarrow \frac{\Phi_0 - \Phi_L}{L}t + u_s(x) = \frac{\Phi_0 - \Phi_L}{L}t + \frac{\Phi_0 - \Phi_L}{2L}x^2 - \Phi_0 x + b,$$

where

$$b = \frac{1}{L} \int_0^L f(x) dx$$

Note that $v(x, t)$ satisfies the heat equation $\partial_t v = \partial_x^2 v$, with homogeneous flux boundary conditions and the initial condition $v(x, 0) = f(x) - u_s(x)$.

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(a)

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \frac{\partial^2}{\partial x^2} u(x, t) + (a + bt) \sin 3x, & 0 \leq x \leq \pi \\ u(0, t) &= u(\pi, t) = 0 \\ u(x, 0) &= 0 \end{aligned}$$

The boundary conditions and initial conditions are all zero. If we do not have the source $Q(x, t) = (a + bt) \sin 3x$, the solution will be the trivial solution $u = 0$. We also note that the source is one of the eigenmodes, thus we expect the solution with the source will be of the same form:

$$u(x, t) = f(t) \sin 3x$$

The boundary conditions $u(0, t) = u(\pi, t) = 0$ are satisfied. Substituting u into the heat equation, we have

$$\begin{aligned} f'(t) \sin 3x &= -9f(t) \sin 3x + (a + bt) \sin 3x \\ \Rightarrow f'(t) + 9f(t) &= a + bt \\ f(0) &= 0 & (\text{from the initial condition } u(x, 0) = 0) \end{aligned}$$

Solving this nonhomogeneous first order ODE, we get

$$f(t) = \frac{1}{81}[(b - 9a)e^{-9t} + 9a - b + 9bt]$$

Thus the solution of the heat flow problem is

$$\boxed{u(x, t) = \frac{1}{81}[(b - 9a)e^{-9t} + 9a - b + 9bt] \sin 3x} \quad (4)$$

(b) We use Green's function to solve the heat equation. Expressing the heat source in eigenfunction expansions, we get

$$\begin{aligned} Q(x, t) &= (a + bt) \sin 3x = \sum_{n=1}^{\infty} q_n(t) \phi_n(x) \\ &= q_3(t) \phi_3(x), \end{aligned}$$

where

$$\begin{aligned} \phi_3(x) &= \sin 3x \\ \lambda_3 &= 9 \\ q_3(t) &= \frac{2}{\pi} \int_0^{\pi} dy \phi_3(y) Q(y, t) = a + bt \end{aligned}$$

So the Green's function for our heat flow problem is

$$G(x, y, t - t') = \frac{2}{\pi} e^{-9(t-t')} \sin(3x) \sin(3y)$$

We then get

$$\begin{aligned} u(x, t) &= \int_0^t dt' \int_0^{\pi} dy G(x, y, t - t') Q(y, t') \\ &= \int_0^t dt' \int_0^{\pi} dy \frac{2}{\pi} e^{-9(t-t')} \sin(3x) \sin(3y) (a + bt') \sin(3y) \\ &= \sin(3x) e^{-9t} \int_0^t dt' e^{9t'} (a + bt') \\ &= \frac{1}{81} [(b - 9a)e^{-9t} + 9a - b + 9bt] \sin 3x \end{aligned}$$

Thus the Green's function formula produces the same solution.