

18.303 Problem Set 2

due: Thursday, October 8, 2009

6. The regular Bessel function $J_n(x)$ sometimes appears through an integral representation

$$J_n(x) = \frac{1}{\pi i^n} \int_0^\pi d\theta e^{ix \cos(\theta)} \cos n\theta.$$

- (a) Use differentiation under the integral and other appropriate steps to show that the function defined by the integral satisfies the Bessel ODE.
(b) Argue that the first n power series coefficients of the function defined by the integral vanish, so that $J_n(x) \sim x^n$ as $x \rightarrow 0$.
7. Consider the Bessel equation of order $1/2$ on the interval $[1, 2]$ with Dirichlet boundary conditions

$$x^2 y'' + xy' - \frac{1}{4}y = -k^2 x^2 y, \quad y(1) = y(2) = 0.$$

- (a) Show that

$$y_1 = \frac{\sin kx}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{\cos kx}{\sqrt{x}}$$

are two independent solutions of the equation.

- (b) Find the eigenvalues and the corresponding eigenfunctions.
(c) Change the boundary conditions to $y(1) = y'(2) = 0$ and show that the eigenvalues are the roots of the equation $\tan k = 4k$.
8. In class we proved results for the reality and sign of eigenvalues in Sturm-Liouville problems for pure boundary conditions. Here you are asked to extend them, if possible to the case of mixed boundary conditions $\phi(a) + k_a \phi'(a) = 0$ and $\phi(b) + k_b \phi'(b) = 0$.
- (a) Show that eigenvalues λ must be real for all k_a, k_b .
(b) Find conditions on k_a, k_b which are sufficient to prove $\lambda > 0$. You may assume that $q(x) \leq 0$.
9. Generalize the Sturm comparison theorem to study problems with general boundary condition $\phi(a) + k_a \phi'(a) = 0$. Show that the eigenfunction $\phi_{n+1}(x)$ has at least one zero between $x = a$ and the first zero of $\phi_n(x)$.

Problem 10 on reverse page.

10. The Gram-Schmidt orthogonalization procedure is a systematic method to start with a (finite) set of *linearly independent* vectors $v_1, v_2, \dots, v_n$ and produce a set of *orthogonal* vectors $e_1, e_2, \dots, e_n$. The orthogonal set has the property $e_k = \sum_{i=1}^k a_{(i,k)} v_i$ with definite coefficients $a_{(i,k)}$. Two convenient on-line references to the procedure are: ** en.wikipedia.org/wiki/GramSchmidt_process, ** and ** www.math.hmc.edu/calculus/tutorials/gramschmidt/gramschmidt.pdf. ** In this problem, students are asked to apply this procedure in two situations, the first for the vector space $\mathbb{R}^3$ and the second for the function space $C[-1, 1]$.

(a) Start with the three independent vectors $v_1 = (1, 0, 0)$, $v_2 = (1, 2, 0)$, $v_3 = (1, 2, 3)$ and produce an orthogonal set e_1, e_2, e_3 with the properties above.

(b) In the function space $C[-1, 1]$, start with the four monomials $v_0 = 1$, $v_1 = x$, $v_2 = x^2$, $v_3 = x^3$. and produce a set of n th order polynomials $P_n(x)$, $n = 0, 1, 2, 3$ which satisfy $P_n(1) = 1$ and the orthogonality condition $\int_{-1}^1 dx P_m(x) P_n(x) = 2\delta_{mn}/(2n+1)$. Check that your answers agree with the standard Legendre polynomials, see page 340 of Haberman. If not fix your mistakes with the Gram-Schmidt procedure. Then show explicitly that $P_2(x)$ and $P_3(x)$ satisfy the Legendre differential equation (for the appropriate value of n).