

18.303 Problem Set 1 – REVISED

due: Thursday, September 24, 2009

1. Consider the heat equation $\partial_t u = k \partial_x^2 u$ on the interval $0 \leq x \leq L$ with the boundary conditions $u(0, t) = 0$ and $\partial_x u(L, t) = 0$. (These conditions correspond to ice water on the left side of a metal bar and a perfect heat insulator on the right side.)

a. Solve the boundary value problem and obtain the separable solutions

$$u_n(x, t) = \sin\left((n + \frac{1}{2})\pi x/L\right) \exp\left(-[(n + \frac{1}{2})\pi/L]^2 kt\right), \quad n = 0, 1, 2, \dots$$

b. Assume, as we will argue later in class, that the following expansion formulas hold for the general solution and the initial data $u(x, 0) = f(x)$:

$$u(x, t) = \sum_{n=0}^{\infty} b_n u_n(x, t)$$
$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left((n + \frac{1}{2})\pi x/L\right) \tag{1}$$

$$b_n = \frac{2}{L} \int_0^L \sin\left((n + \frac{1}{2})\pi x/L\right) f(x) dx \tag{2}$$

Assume that $L = \pi$. Find the expansion coefficients b_n for the initial temperature distribution $f(x) = x$. How fast do they fall as $n \rightarrow \infty$?

c. Prepare plots similar to those in the class notes (page 9.5) to illustrate the behavior of the truncated expansion for $f(x) = x$. Truncate with one ($N=0$), two ($N=1$), and three ($N=2$) modes. Briefly explain why the truncations with so few modes are accurate for this case but not for the problem considered in class.

2. Find the transformation of y which takes the general second order linear, homogeneous ODE

$$\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0$$

into an equation of the form

$$\frac{d^2 u}{dt^2} + q(t)u = 0.$$

Also, find how the function $q(t)$ depends on $P(t)$ and $Q(t)$.

(*Hint:* Write $y(t) = u(t)v(t)$ and then choose the function $v(t)$ in such a way that the coefficient of du/dt cancels.)

3. *Complex Exponentials*

Find the solution of the initial value problem

$$9y'' + 6y' + 82y = 0, \quad \text{with } y(0) = -1, y'(0) = 2$$

by writing the basis functions which span the solution space as complex functions.

4. *Reduction of Order and Variation of Parameters*

Consider the equation

$$x^2y'' - 4xy' + 6y = x^4 \sin x$$

and observe that $y_1 = x^2$ is a solution of the associated homogeneous equation. Use reduction of order to find the second, linearly independent solution of the homogeneous equation and then use the method of variation of parameters to derive the general solution to the above equation.

5. *Green's function*

This problem concerns the Green's function solution of the ODE

$$t^2y'' + 2ty' - 2y = f(t)$$

(where we use $d/dt = '$ to simplify the typing.)

a. Find the Green's function $G(t, s)$ by solving the appropriate IVP which was discussed in class. Compare your answer with the Green's function obtained by the MVP method. They should agree!

b. Use the Green formula

$$y(t) = \int_{t_0}^t G(t, s)R(s)$$

to solve the inhomogeneous equation for source $f(t) = t^k$ assuming that $t_0 > 0$. Compare the solution with that obtained by the method of undetermined coefficients from 18.03. The difference between the two solutions should be a combination of homogeneous solutions provided that you specify $R(t)$ properly.