

Ex: Show for any cr. $\lambda = e^{i\theta} |\lambda|$, ϕ_{gen} violates 82
the growth cond unless phase $\theta = 0$

2i) So we are left with $\lambda = k^2 > 0$ and we find soln
 $\phi_k(x) = e^{ckx}$ $\phi_{-k}(x) = e^{-ckx}$

These eigenfn's satisfy the growth conditions. Import. change
from all prev prob's in finite regions. All values
of k $-\infty < k < \infty$ are allowed.

Eigen Problems in ∞ domains typically have continuous spectrum
This is the major new feature we must try to understand.

Proceed by analogy with heat flow in $[-L, L]$

a) Sep. sol's of heat eq are $u_k(x, t) = e^{-k^2 t} e^{ckx}$ $-\infty < k < \infty$

b) Linearity suggests that contin. superp. of these is also a soln.

$$u(x, t) = \int_{-\infty}^{\infty} dk c(k) e^{-k^2 t} e^{ckx}$$

↑ note 'rapid convergence' at large k

c) At $t=0$ we must satisfy IC

$$\lim_{t \rightarrow 0^+} u(x, t) = f(x) = \int_{-\infty}^{\infty} dk c(k) e^{ckx}$$

Our prob will be solved if it is true that every well behaved
 $f(x)$ on $\mathbb{R}$ can be represented as a contin. superp. of
plane waves and how to obtain $c(k)$ from $f(x)$.

This representation is the Fourier transform

Students should read Sec 10.3.1 of H in which F.trf is
obtained as limit of F. series on $[-L, L]$.

In lecture we will simply state the ^{conclusion} carefully:

Thm: If $f(x)$ is piecewise smooth on $\mathbb{R}$ and $\int_{-\infty}^{\infty} dx |f(x)|$
exists, then $f(x)$ can be represented as a

Fourier integral: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ckx} F(k)$ where dk

F.trf of $f(x)$ is $F(k) = \int_{-\infty}^{\infty} dx e^{-ckx} f(x)$

i) $f(x)$ is a contin. superp. of plane waves.

ii) all info on $f(x)$ is encoded in its transform $F(k)$.

Class conventions are different from Haberman.

$$F(k)_{\text{class}} = 2\pi F(-\omega)_{\text{text}} \quad k = -\omega$$

By subst. + exchange of order $\int dk \leftrightarrow dk$ we write

$$f(x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik(x-y)} f(y) \quad \text{true for any } f(x) \text{ which satisfies conditions of Thm.}$$

Hence we derive the completeness relation of plane waves in the continuum.

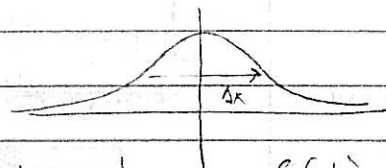
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-y)} = \delta(x-y)$$

Same 'structure' as $\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-y)} = \delta(x-y)$

Examples to illustrate practical work with FT:

1. important practical example Gaussian:

$$f(x) = e^{-\beta x^2}$$



Supersmooth fn with peak of width $\Delta x = \frac{1}{\sqrt{\beta}}$ $f\left(\frac{1}{\sqrt{\beta}}\right) = \frac{1}{e}$

Area $\int_{-\infty}^{\infty} dx e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$

We want to calculate $F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} e^{-\beta x^2}$

Basic result of 18.04: Cauchy's Theorem: if $f(z)$ is an analytic fn of $z = x + iy$ on any closed curve C , then $\oint_C dz f(z) = 0$

Illustrate this important result by choosing $C =$ disc of radius r_0

By def. An analytic function in C has a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where radius of conv. is } R > r_0$$

One example is $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} (x + iy)^n$

We write this to convince you that everything is well defined

We use e^z as an example which should be familiar to you, but the

definition of an analytic is that it has a conv. power series.

closed disc

$$\begin{aligned} \oint_C dz z^n &= i \int_0^{2\pi} r_0 e^{i\theta} d\theta r_0^n e^{in\theta} \\ &= i r_0^{n+1} \int_0^{2\pi} d\theta e^{i(n+1)\theta} = 0 \end{aligned}$$

Note: on circle $dz = d(r_0 e^{i\theta}) = i r_0 e^{i\theta} d\theta$

Hence $\oint \frac{dz f(z)}{z^n} = 0$ if $f(z)$ is analytic in the disc.

Ex for -ve powers (which means $f(z)$ has a pole at $z=0$)

$$\oint dz z^{-n} = r_0^{1-n} \int_0^{2\pi} d\theta e^{i(1-n)\theta}$$

we will need this later in lecture

$$= \begin{cases} 0 & \text{if } n > 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$



Please accept $\oint dz f(z) = 0$ is true for any curve C

We will now compute the FT: $F(k) = \int_{-\infty}^{\infty} dx e^{-ikx} e^{-\beta x^2}$

Use important methods: a) complete the square b) Cauchy Thm

Step 1: $F(k) = e^{-k^2/4\beta} \int_{-\infty}^{\infty} dx e^{-\beta(x + \frac{ik}{2\beta})^2}$ Sec 10.3 of H

Step 2: If shift $r = \frac{ik}{2\beta}$ were real, we would change variables to $z = x + r$, the integral would be $\int_{-\infty}^{\infty} dz e^{-\beta z^2} = \sqrt{\frac{\pi}{\beta}}$

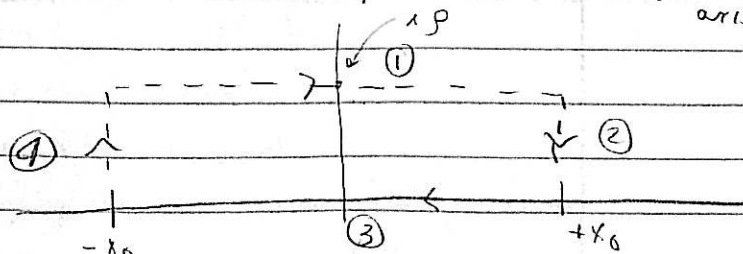
Then we would find $F(k) = \sqrt{\frac{\pi}{\beta}} e^{-k^2/4\beta}$. ← This will turn out to be the correct answer, but...

Step 3. We need to be careful because $r = \frac{ik}{2\beta} = \frac{k\beta}{2\beta}$ is imaginary, so shifted integ. is along cr. line

$$I = \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} dz e^{-\beta z^2}$$

We will use Cauchy's Thm to show that integral is same as on real axis.

Cauchy's Thm tells us that integ over entire contour $= 0$, b/c $e^{-\beta z^2}$ is analytic everywhere.



$$\int dz e^{-\beta z^2} = 0$$

(1) + (2) + (3) + (4)

We will show that

$$\lim_{x_0 \rightarrow \infty} \int_{(2)} dz e^{-\beta z^2} = 0 \quad \text{Same for } \int_{(4)}$$

Then $I = + \lim_{x_0 \rightarrow \infty} \int_{(1)} = - \int_{(2)} = + \int_{-\infty}^{\infty} dz e^{-\beta z^2} = \sqrt{\frac{\pi}{\beta}}$

$\uparrow$
 sign change b/c of direction of contour.

To prove $\checkmark$, look at

$$e^{-\beta z^2} = e^{-\beta(x+iy)^2} = e^{-\beta x^2} e^{\beta y^2} e^{-2i\beta xy}$$

$$\int_{(2)} dz e^{-\beta z^2} = -i e^{-\beta x_0^2} \int_0^P dy e^{\beta y^2} e^{-2i\beta x_0 y}$$

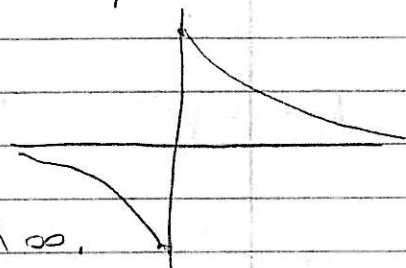
$$\text{So } \left| \int_{(2)} \right| \leq e^{-\beta x_0^2} \int_0^P dy e^{\beta y^2} \xrightarrow{0 \text{ as } x_0 \rightarrow \infty} 0$$

So we are done. Conclusion: FT of $e^{-\beta x^2}$ is $F(k) = \sqrt{\frac{\pi}{\beta}} e^{-k^2/4\beta}$

FT of a Gaussian of width $\Delta x = \frac{1}{\sqrt{\beta}}$ in x space is another Gaussian of width $\Delta k = \sqrt{4\beta}$ in k -space.

- i) Hence $\Delta x \Delta k = 2 \leftarrow$ example of uncertainty principle in q.m.
- ii) Note $f(x)$ is super smooth in x , so $F(k)$ falls off superfast at large $|k|$.

2, Another example: $f(x) = \begin{cases} e^{-ax} & x > 0 \\ -e^{ax} & x < 0 \end{cases}$



An odd fn., jump disc at $x=0$, damped at ∞ .

The FT is

$$F(k) = \left\{ \int_0^{\infty} dx e^{-(a+ik)x} - \int_{-\infty}^0 dx e^{(a-ik)x} \right\}$$

$$= \frac{1}{a+ik} - \frac{1}{a-ik} = -\frac{2ik}{a^2+k^2} \quad \text{we learn that}$$

i) FT of a real $f(x)$ need not be real

Ex: FT of a real even $f(x)$ is real

FT of a real odd $f(x)$ is pure imaginary

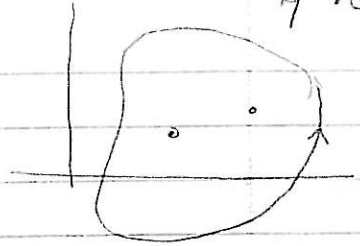
ii) $F(k) \sim \frac{1}{k}$ for large k , as expected from jump at $x=0$

How would we verify the F.T. formulae in this case

$$\int_{-\infty}^{\infty} dx e^{ikx} F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} F(k) \quad F(k) = \frac{1}{a+ik} - \frac{1}{a-ik}$$

Use Cauchy residue formula, another imp. practical result of 18.04.

Describe a fn $g(z)$ which is holomorphic inside a closed curve C except for simple poles at z_1 and z_2 with residues r_1 and r_2 .



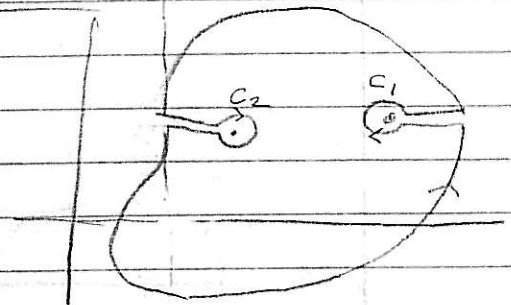
$g(z) = \frac{r_1}{z-z_1} + \frac{r_2}{z-z_2} + h(z)$ where $h(z)$ is holomorphic (or analytic). This means that $h(z)$ has a conv. power series in C .

Claim:

$$\frac{1}{2\pi i} \oint_C g(z) dz = r_1 + r_2$$

Outline of pf: i) Inside C' $g(z)$ is holomorphic, so

$$\oint_{C'} g(z) dz = 0$$



ii) This remains true in limit

a limit as the two narrow channels collapse and $C' \rightarrow C \cup C_1 \cup C_2$

Hence

$$0 = \oint_{C'} g(z) dz = \oint_C g(z) dz - \int_{C_1} g(z) dz - \int_{C_2} g(z) dz$$

C_1 and C_2 are small closed circles around z_1 and z_2 - sign because the $\oint$ is defined using ccw orientation.

So

$$\oint_C g(z) dz = \oint_{C_1} g(z) dz + \oint_{C_2} g(z) dz$$

iii) Consider $\oint_{C_1} g(z) dz = \oint_{C_1} dw \left\{ \frac{r_1}{w-z_1} + \tilde{h}(w) \right\} = 2\pi i r_1$

We can use our work in p 85 b/c $\tilde{h}(w) = \sum_{n=0}^{\infty} c_n w^n$ a conv. power series around C_1

Similarly $\oint_{C_2} g(z) dz = 2\pi i r_2$ and we have

proved $\square$. Note: a more general form of Cauchy formula allows for series of poles up to order N .

If $g(z) = \sum_{j=1}^N \frac{r_j}{(z - z_0)^j} + h(z)$

Then $\frac{1}{2\pi i} \oint_C g(z) dz = \begin{cases} r_1 & \text{if } z_0 \text{ is inside } C \\ 0 & \text{if } z_0 \text{ is outside } C \end{cases}$

In our applic: $g(z) \rightarrow \frac{-i e^{kx}}{k - ia} + \frac{-i e^{kx}}{k + ia}$
 $z \rightarrow k$

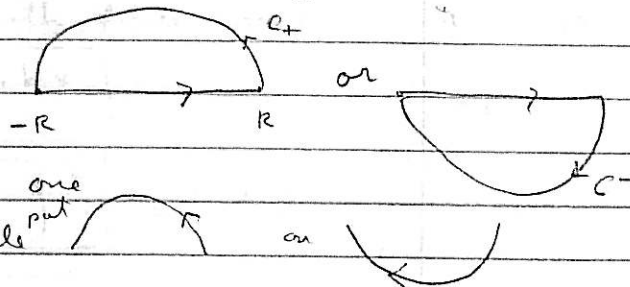
poles at $k = ia$ residue $-i e^{-ax}$
 $k = -ia$ " $-i e^{ax}$

The FT we want is

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{-R}^R dk \left[\frac{e^{ckx}}{k - ia} + \frac{e^{ckx}}{k + ia} \right]$$

We need a closed contour to apply Cauchy formula.

Correct choice of contour is the one for which the semi-circle vanishes as $R \rightarrow \infty$.



Let $k = k_R + i k_I$, $e^{ckx} = e^{-k_I x} e^{ck_R x}$.

Behavior of $\int_{\gamma} dk F(k) e^{ckx}$ is determined by what happens on γ as $\gamma \rightarrow \infty$.

$k_I \rightarrow \infty$. It is not hard to see that

a) if $x > 0$ e^{ckx} is damped in upper $\frac{1}{2}$ plane - choose C^+

b) if $x < 0$ " " " " lower " " - choose C^- .

$$\begin{aligned} \text{a) } \frac{1}{2\pi i} \oint_{C^+} dk F(k) e^{ckx} &= \frac{1}{2\pi i} \oint_{C^+} dk e^{ckx} \left[\frac{1}{k - ia} + \frac{1}{k + ia} \right] \\ &= e^{ckx} \Big|_{k=ia} + 0 \quad \uparrow \text{pole is outside } C^+ \\ &= e^{-ax} \end{aligned}$$

So as $R \rightarrow \infty$

$$\oint_{C^+} \rightarrow \int_{-\infty}^{\infty} + \text{arc} \rightarrow 0$$

Then $\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk F(k) e^{ckx} = e^{-ax}$ correct

$$b) \frac{1}{2\pi} \int_{c-} dk F(k) e^{ikx} = \frac{1}{2\pi i} \int_{c-} dk e^{ikx} \left[\frac{1}{k-ia} - \frac{1}{k+ia} \right]$$

On the nose

$$\begin{array}{c} \text{O} \\ \parallel \\ 0 \end{array} \quad \begin{array}{c} \downarrow \\ -e^{ikx} \big|_{k=-ia} \end{array}$$

$$= -e^{ikx}$$

direction of contour



Summary: $\frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} F(k) = \begin{cases} e^{-ax} & x > 0 \\ -e^{ax} & x < 0 \end{cases}$

The basic FT then works in this case.

Comments: i) it is interesting that $\text{sgn } x$ determines the contour.
ii) ex. integ. in freq. very useful for F. transf.

Given these examples we can now go back to apply to heat flow on $\mathbb{R}$ on p 82. By analogy with finite intervals, we would

$$u(x,t) = \int_{-\infty}^{\infty} dk c(k) e^{-k^2 t} e^{ikx} \quad c(k) = \frac{1}{2\pi} F(k)$$

We satisfy II if $c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{-iky} f(x).$

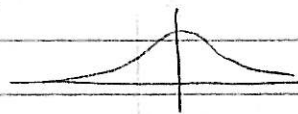
These formulas give a complete set of heat flow prob on $\mathbb{R}$ for any reasonable initial temp. $f(x).$

$$\Delta x = \frac{1}{\sqrt{\beta}}$$

As an example, supp. initial temp is Gaussian $f(x) = e^{-\beta x^2}$

What is the future evolution of temp $u(x,t)$?

We already know the FT: $F(k) = \sqrt{\frac{\pi}{\beta}} e^{-k^2/4\beta}$



So the temp at future time t is

$$u(x,t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int_{-\infty}^{\infty} dk e^{-\frac{k^2}{4\beta}} e^{-k^2 t} e^{ikx}$$

There is another Gaussian integral we can do and will do:

Let $\frac{1}{4\beta(t)} = \frac{1}{4\beta} + t$ Ex: $\beta(t) = \frac{\beta}{1+4\beta t}$

$$u(x,t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int_{-\infty}^{\infty} dk e^{-\frac{k^2}{4\beta(t)}} e^{ikx} = \sqrt{\frac{\beta(t)}{\beta}} e^{-\beta(t)x^2} = \frac{1}{\sqrt{1+4\beta t}} e^{-\beta x^2/(1+4\beta t)}$$

Integ uses prev result on p 86

with $\beta \rightarrow \frac{1}{4\beta(t)}$

$$= \frac{1}{\sqrt{1+4\beta t}} e^{-\beta x^2/(1+4\beta t)}$$

Final result

Interp: At time t , the width of the dist. is

$$\Delta x_t = \sqrt{\frac{1+4Dt}{\beta}} \Rightarrow \text{The peak broadens in time at rate } t^{1/2}$$

which agrees with gen'l relation of scales $\frac{1}{\sqrt{t}}$ in diffusion.

Height at max: $u(0,t) = \frac{1}{\sqrt{1+4Dt}}$ decreases with time.

Ex: Show that total thermal energy is constant in t

$$\int_{-\infty}^{\infty} dx u(x,t) = \int_{-\infty}^{\infty} dx u(x,0) = \int_{-\infty}^{\infty} dx e^{-\beta x^2} = \sqrt{\frac{\pi}{\beta}}$$

It can be shown that this is true for a general IC. $u(x,0) = f(x)$ provided $\int_{-\infty}^{\infty} dx f(x)$ is finite.

Green's fn approach to the same prob: We would like to express

$$sol as \quad u(x,t) = \int_{-\infty}^{\infty} dy G(x,y,t) f(y)$$

where G encodes universal dynamics of diffusion on $\mathbb{R}$ and only $f(y)$ changes from prob. to prob. One way to

proceed is to substitute $c(k)$ into formula for $u(x,t)$ and exch. $\int dk \leftrightarrow \int dy$. This works, but it is an old and boring method. New approach $\rightarrow$ just

Define: $G(x,y,t)$ as sol to the heat eqn for the IC $f(x) \rightarrow \delta(x-y)$ $\leftarrow$ initial spks of particles at $x=y$

Apply prev. formulas $\boxed{\frac{1}{2\pi}}$

$$c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{-ikz} f(z) \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{-ikz} \delta(z-y) = \frac{e^{-iky}}{2\pi}$$

$$\text{Then } u(x,t) \rightarrow \left[G(x,y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-k^2 t} e^{ik(x-y)} \right]$$



This is the Gr. fn for heat flow on $\mathbb{R}$. Comments:

i) same spectral form as in prev situation

$$G(x,y,t) = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n t}}{N_n^2} \phi_n(x) \phi_n(y) \quad \text{but with } \sum_n \rightarrow \int dk$$

ii) trans. inv: $G(x,y,t) = G(x-y,t)$. True because the PDE $\partial_t u = \partial_x^2 u$ describes heat flow on homogeneous region and there are no physical BC's such as those on $[0,\pi]$

For D BC's on $[0, \pi]$ we found

$$G(x, y, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny = \frac{1}{\pi} \sum_n e^{-n^2 t} \left[\cos n(x-y) - \cos n(x+y) \right]$$

$\uparrow$ non-inv

(ii) The $\int dk$ in $\boxed{}$ is a special case of our Gaussian $e^{-\beta x^2} = \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \int dk e^{-\frac{k^2}{4\beta}} e^{ikx}$

We identify $t = \frac{1}{4\beta}$ $\beta = \frac{1}{4t}$ and can write

$$G(x-y, t) = \sqrt{\frac{\beta}{\pi}} e^{-\beta(x-y)^2} = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$$

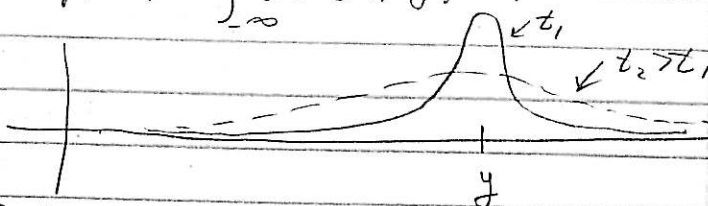
explicit

We have found the Gr. fn for heat flow on $\mathbb{R}$ in closed form!
It is also called the heat kernel or fund sol of the heat eq

Two Exercises: Ex 1 Check explicitly that $(\partial_t - \partial_x^2) G(x-y, t) = 0$

Ex 2: $\int_{-\infty}^{\infty} dx G(x-y, t) = \int_{-\infty}^{\infty} dx \delta(x-y) = 1$

Plot



Scales $(x-y)^2/t$ as expected:

Using the heat kernel, we can write 'sol. of general' IVP as

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} dy e^{-(x-y)^2/4t} f(y)$$

The same G can be used to find sol of heat flow with any source:
 $\partial_t u(x, t) = \partial_x^2 u(x, t) + Q(x, t)$

Recall sol of same prob. for heat flow on $-[0, \pi]$ p 52

$$u(x, t) = \int_0^t dt' \int_0^\pi dy G(x, y, t-t') Q(y, t')$$

$$Q_n = \sin nx$$

We derived it by expanding u and Q in eigenfn expans.

We could do the analogous here using Fourier tfs of u or Q . But it is much simpler to write down a formula of some structure and then verify it.

$$u(x, t) = \int_0^t dt' \int_{-\infty}^{\infty} dy G(x-y, t-t') Q(y, t')$$

Interp: Temp at pts x and time t depends on action of source for all earlier times $t' < t$ and all source pts y .

Verify this formula:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u = \int_0^t dt' \int dy \overbrace{(\partial_t - \partial_x^2) G(x-y, t-t')}^{=0} Q(y, t')$$

$$+ \lim_{t' \rightarrow t^-} \int dy G(x-y, t-t') Q(y, t) \quad \text{done}$$

$$\hookrightarrow \int dy \delta(x-y) Q(y, t) = Q(x, t)$$

So we have indeed found sol of heat flow with sources on $\mathbb{R}$ (by a much faster method than the eigenfn exp. = FT in this case).

Moral of the story: many linear PDE probs. have a common "structure". If you recognize the structure of a problem, you can save a lot of time in solving it.

New (but related) problem:

Heat flow on half-line: $0 \leq x < \infty = \mathbb{R}^+$. The prob is

$$\partial_t u = \partial_x^2 u \quad x \in \mathbb{R}^+$$

$$\text{IC } u(x, 0) = f(x).$$

$$\text{BC } u(x, t) \text{ bounded as } x \rightarrow \infty$$

$$\text{D } u(0, t) = 0$$

$$\text{or } \text{N } \partial_x u(0, t) = 0$$

One way to proceed is by sep of variables and expansion in eigenfn for D prob $\sin kx$ or N prob. $\cos kx$.

(This would involve Fourier sine or cosine trf of Sec 10.5 of H)

Instead we will use method of image charges, which is much quicker when applicable. We will next to construct

Green's fns $G_D(x, y, t)$ and $G_N(x, y, t)$ directly:

These Green's fns must satisfy:

$$1) (\partial_t - \partial_x^2) G(x, y, t) = 0 \quad \text{both D and N}$$

$$2) \text{ Evaluation from pt. source } G(x, y, t) \xrightarrow[t \rightarrow 0^+]{\delta(x-y)} \quad \text{D and N}$$

$$3) \text{ BGD } G_D(x=0, y, t) = 0$$

$$\text{or } \text{N } \partial_x G_N(x=0, y, t) = 0$$

If we succeed, the sol of full IVP will be

$$* \quad u(x, t) = \int_0^\infty dy \underbrace{G_D(x, y, t)}_{\text{or } G_N} f(y)$$

← universal structure as in all heat flow prob.

To solve this prob., consider our new friend

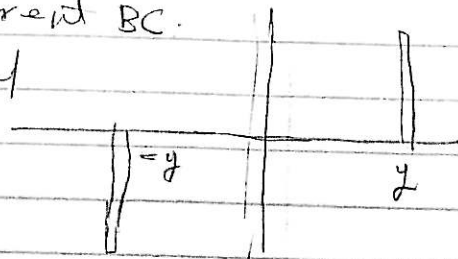
$$G(x-y, t) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \leftarrow \text{Green fun in } \mathbb{R}$$

i) This satisfies everything for $\mathbb{R}^+$ except BC.

ii) Suppose we add fictitious source of particles at $-y$

The future evolution of both sources is

$$G_D(x, y, t) = G(x-y, t) - G(x+y, t)$$



a) This satisfies PDE in t, x since each term does.

b) $G_D(0, y, t) = 0$ since $G(-y, t) = G(y, t)$

So Dirichlet BC is satisfied for all t .

c) As $t \rightarrow 0^+$ $G_D(x, y, t) \rightarrow \delta(x-y) - \delta(x+y)$

It seems to have an extra source $\uparrow$, but we are going to use G_D in x only in region $x > 0$ $y > 0$.

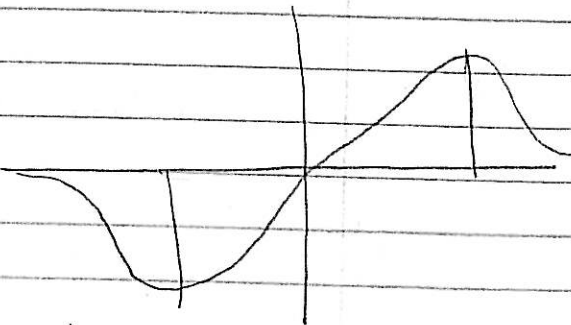
So peak from $\delta(x+y)$ never occurs, and we can drop it

Conclusion $G_D(x, y, t)$ is correct D.G. for $\mathbb{R}^+$ -line problem

Ex: $G_D(x, y, t)$ is odd in x

$$G_D(x, y, t) = -G_D(-x, y, t)$$

Hence $G_D(0, y, t) = 0$ automatically



For Neumann problem we add the image ~~same~~ term.

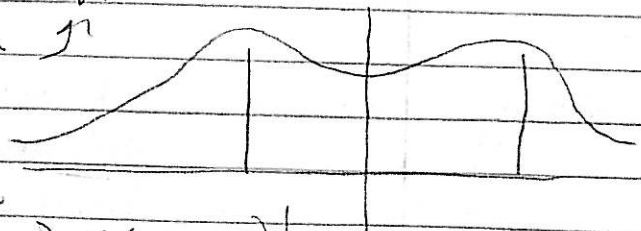
$$G_N(x, y, t) = G(x-y, t) + G(x+y, t)$$

direct term $\uparrow$ image term $\uparrow$

This is an even fn of x ,

which automatically satisfies

$$\left. \frac{\partial G_N}{\partial x}(x, y, t) \right|_{x=0} = 0$$



Ex: Use spectral repn of $G(x \pm y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-k^2 t} e^{ik(x \pm y)}$

Remark

$$G_N(x, y, t) = \frac{2}{\pi} \int_0^{\infty} dk e^{-k^2 t} \cos kx \cos ky$$

This is the form expected from an expansion in the Fourier/Neumann modes $\phi_k(x) \sim \cos kx$ on $\mathbb{R}^+$

You can see that image charge methods are very powerful when they work, they typically work only when there is a reflection sym. in the problem, in this case in $x \rightarrow -x$.

Ex: $G_{D,N}(x,y,t) = G_{D,N}(y,x,t)$ symmetry or reciprocity

The Green's fn $G_{D,N}$ can be used to solve the following probs on $\mathbb{R}^+$ line:

1. IVP $u(x,t) = \int_0^\infty dy G_{D,N}(x,y,t) f(y)$

2. Source $u(x,t) = \int_0^t dt' \int_0^\infty dy G_{D,N}(x,y,t-t') Q(y,t')$

↙ In both cases we have simply written formulas of the same structure as in previous cases.

3. third physical possibility is to put an inhom. B.C. at $x=0$

$$\begin{array}{l} \partial_t u(x,t) = \partial_x^2 u(x,t) \quad x > 0 \\ D \quad u(0,t) = A(t) \\ N \text{ or } \partial_x u(0,t) = -\Phi(t) \end{array} \quad \begin{array}{c} t \uparrow \\ A(t) \end{array}$$

We have not yet considered analogous prob for a finite interval:

$$\begin{array}{l} \partial_t u(x,t) = \partial_x^2 u(x,t) \quad x \in [0,\pi] \\ u(0,t) = A(t) \quad \leftarrow \text{time dep. heat} \\ u(\pi,t) = B(t) \quad \leftarrow \text{baths at endpoints of a metal bar.} \end{array} \quad \begin{array}{c} t \\ A(t) \end{array} \quad \begin{array}{c} B(t) \\ x \rightarrow 0 \end{array}$$

$$u(x,0) = f(x) \rightarrow 0 \quad \leftarrow \text{trivial IC, because we already know how to solve IVP}$$

Strategy:

1. Solve finite interval prob using eigenfn. expansion $\rightarrow$ Sec 8.4 of H
2. Express sol using $G(x,y,t)$ for the finite interval (an old friend)
3. Write formula of same "structure" to solve prob on $\mathbb{R}^+$

Reminder: eigenfn. for prob on $[0,\pi]$ are

$$\phi_n(x) = \sin nx$$

$$n = 1, 2, 3, \dots \quad \lambda_n = n^2$$

$$\begin{aligned} (\phi_n, \phi_m) &= \int_0^\pi dx \sin nx \sin mx \\ &= N_n^2 \delta_{nm} = \frac{\pi}{2} \delta_{nm} \end{aligned}$$