

1. Uniqueness for sol of Δ/D prob in V :

$$\nabla^2 u = 0 \quad \vec{x} \in V \quad u|_{\partial V} = f(\vec{x})$$

Let $u_1(\vec{x})$ and $u_2(\vec{x})$ be two sol., Consider $v = u_1 - u_2$.

$$\text{Then } \nabla^2 v = 0 \quad \vec{x} \in V \quad v|_{\partial V} = 0$$

$$\text{Green, } \int_V d^3x [\underbrace{\vec{\nabla} v \cdot \vec{\nabla} v}_{=0} + \underbrace{v \nabla^2 v}_{=0}] = \int_{\partial V} d^2x \underbrace{v}_{=0} \hat{n} \cdot \vec{\nabla} v$$

Thus $\vec{\nabla} v(\vec{x}) = 0 \quad \vec{x} \in V \Rightarrow v = \text{const.}$ But only $v(\vec{x}) \equiv 0$ satisfies BC, Hence $u_1(\vec{x}) \equiv u_2(\vec{x})$ Unique Sol.

2 Δ/N problem: $\nabla^2 \vec{u}(\vec{x}) = 0 \quad \vec{x} \in V \quad \hat{n} \cdot \vec{\nabla} u|_{\partial V} = \Phi(\vec{x})$ flux.

A necessary cond. for existence of any sol is that

$$\text{solubility cond. (p 64-65 of notes) is satisfied } \int_{\partial V} d^2x \Phi(\vec{x}) = 0$$

Then $v = u_1(\vec{x}) - u_2(\vec{x})$ satisfies

$$\nabla^2 v = 0 \quad \text{in } V \quad \text{and } \hat{n} \cdot \vec{\nabla} v|_{\partial V} = 0 \quad \text{on } \partial V$$

$$\text{Green, } \int_V d^3x [\underbrace{\vec{\nabla} v \cdot \vec{\nabla} v}_{=0} + \underbrace{v \nabla^2 v}_{=0}] = \int_{\partial V} d^2x \underbrace{v}_{=0} \hat{n} \cdot \vec{\nabla} v$$

Again $\vec{\nabla} v \equiv 0 \Rightarrow v(\vec{x}) = c.$ ← But this satisfies Neumann BC

Conclusion: (i) a sol of Δ/N prob. can exist only if $\int_{\partial V} d^2x \Phi = 0$.

(ii) if u_1 and u_2 are any two sol., then $u_2(\vec{x}) = u_1(\vec{x}) + c.$

unique up to additive const.

Green's Fm for Laplace operator: Sec 9.5 of H.

Consider Poisson's Eq: $\nabla^2 u(\vec{x}) = \rho(\vec{x})$ in V $\left\{ \begin{array}{l} \rho \text{ is source,} \\ \rho = \text{charge density} \\ \text{in source for} \\ \text{electric potential } V \end{array} \right.$

$$u(\vec{x})|_{\partial V} = 0$$

Suppose we define Green's fn $G(\vec{x}, \vec{y})$ as sol for a pt. source at $\vec{x} = \vec{y}$:

$$* \text{ Hence } \nabla_{\vec{x}}^2 G(\vec{x}, \vec{y}) = \delta^3(\vec{x} - \vec{y}) \quad G(\vec{x}, \vec{y})|_{\vec{x} \in \partial V} = 0$$

$$** \text{ Then we can write } u(\vec{x}) = \int_V d^3x G(\vec{x}, \vec{y}) \rho(\vec{y})$$

and

$$\nabla^2 u(\vec{x}) = \int_V d^3y \nabla_x^2 G(\vec{x}, \vec{y}) \rho(\vec{y}) = \int_V d^3y \delta^3(\vec{x} - \vec{y}) \rho(\vec{y}) = \rho(\vec{x})$$

If $G(\vec{x}, \vec{y})$ is well defined as a sol. of problem $\star$, then $\star\star$ gives "universal" (and unique) sol. of Poisson eqn. ("universal" means valid for any source!)

Interp. $\vec{x}$ is the pt at which we measure $u(\vec{x})$, $\vec{y}$ = source pt.

How can we get a better understanding of $G(\vec{x}, \vec{y})$?

Proceed indirectly via $\mathcal{L}/D$ eigenvalue prob.

$$\nabla^2 \phi_n(\vec{x}) = -\lambda_n \phi_n(\vec{x}) \text{ in } V \quad \phi_n(\vec{x})|_{\partial V} = 0$$

On p 60 we stated key prop of completeness of the $\mathcal{L}/D$ eigenfns $\{\phi_n(\vec{x})\}_{n=1,2,\dots}$:

Any smooth $f(\vec{x})$ has convergent expansion:

$$f(\vec{x}) = \sum_n a_n \phi_n(\vec{x}) \text{ with expn. coeff } a_n = \frac{1}{N_n^2} \int_V d^3y \phi_n(\vec{y}) f(\vec{y})$$

Use our old trick of subst and exch

$$a_n \leftrightarrow \int d\vec{y}$$

$$f(\vec{x}) = \int_V d^3y \sum_n \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} f(\vec{y})$$

$\star\star\star$

$$\text{This suggests } \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} = \delta^3(\vec{x} - \vec{y})$$

This important property is (non-rigorous) expression of completeness of the set $\{\phi_n(\vec{x})\}$. called closure relation in $\mathcal{G}^2, m$

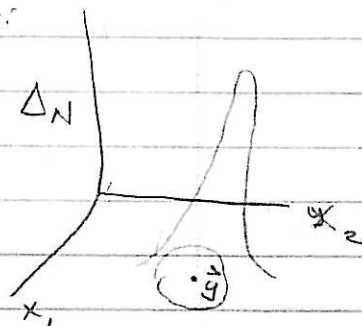
Recall our earlier statement that $\delta^3(\vec{x} - \vec{y})$ always occurs as limit of a family of smooth fns, which become more and more sharply peaked around $\vec{x} = \vec{y}$ in the limit.

In this case, the family of smooth fns is obtained by cutting off the eigenfn. sum at a large value of N ,

$$\Delta_N(\vec{x}, \vec{y}) = \sum_{n=1}^N \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2}$$

$\Delta_N(\vec{x}, \vec{y})$ is a finite sum, it is well defined and is a smooth fn of $\vec{x}$ (or $\vec{y}$). Claim it is more and more sharply peaked as N increases.

Sketch of 2 dim version:



Hence $\lim_{N \rightarrow \infty} \Delta_N(\vec{x}, \vec{y}) = \delta^2(\vec{x} - \vec{y})$

We will exhibit then δ -approx. property of Δ_N in a 1-dim example soon.

Use completeness prop *** to suggest an expression for $G(\vec{x}, \vec{y})$ as an eigenfn. expansion:

Suppose we write candidate expression

$$G(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2}$$

Series converges for $|\vec{x} - \vec{y}| > \delta$, but $G(\vec{x}, \vec{y})$ is sing. at $\vec{x} = \vec{y}$ as we will see

Note: since $\lambda_n \rightarrow \infty$ as n increases, this series converges faster than the completeness rel. ***, so it should be well defined.

Note also:

$$\lim_{x \rightarrow \partial V} G(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} \frac{\lim_{x \rightarrow \partial V} \phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2} = 0$$

(assuming that interchange $\lim_{x \rightarrow \partial V} \leftrightarrow \sum_n$ is valid)

Finally try

$$\nabla_x^2 G(\vec{x}, \vec{y}) = \sum_n \frac{\nabla_n^2 \phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2}$$

$$= \sum_n \frac{-\lambda_n \phi_n(\vec{x}) \phi_n(\vec{y})}{-\lambda_n N_n^2} = \sum_n \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} = \delta^2(\vec{x} - \vec{y})$$

So [] provides an explicit construction of G in terms of the eigenfn's $\{\phi_n(\vec{x})\}$ which satisfies all properties of G (if the interchanges are valid which can be justified)

In fact [] is a universally valid expression for $G(\vec{x}, \vec{y})$ which is called the spectral rep of the Green's fn for the $\mathbb{Z}/D$ problem.

Note symmetry $G(\vec{x}, \vec{y}) = G(\vec{y}, \vec{x})$ called reciprocity in H.

We now have some understanding that G exists and is well defined. We also know how to use it to solve the Poisson eq $\nabla^2 u = p \Rightarrow u(\vec{x}) = \int_V d^3y G(\vec{x}, \vec{y}) p(\vec{y})$

Now show how to use G to solve bdy value prob. — Sec 9.5.5

$$\nabla^2 u = 0 \text{ in } V \quad u(\vec{x}) \Big|_{\partial V} = f(\vec{x})$$

Use Green's (with $\vec{x} \rightarrow \vec{y}$)

$$\int_V d^3y [u(y) \nabla^2 v(y) - v(y) \nabla^2 u(y)] = \int_{\partial V} d^2y [u(y) \hat{n} \cdot \nabla v(y) - v(y) \hat{n} \cdot \nabla u(y)]$$

Choose $u(\vec{y})$ as closed sol of PDE.

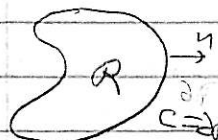
Choose $v(y) = G(x, y)$ ($\vec{x}$ is a spectator!)

$$\text{l.h.s} = u(x) = \text{r.h.s} = \int_{\partial V} d^2y \hat{n} \cdot \nabla_y G(\vec{x}, \vec{y}) f(\vec{y})$$

This is gen'l form of the sol. Universal for any bdy data.

If we define $g(\vec{x}, \vec{y}) = \hat{n} \cdot \nabla_y G(\vec{x}, \vec{y})$, we get +

$$u(\vec{x}) = \int_{\partial V} d^2y g(\vec{x}, \vec{y}) f(\vec{y})$$



For a 2-dim region R with bdy $C = \partial R$, our formula would be

$$u(\vec{x}) = \int_C dy g(\vec{x}, y) f(y) \quad \text{with } y \text{ a coord in bdy}$$

This is the extension of Poisson integral formula on p 70 to a general region R . Therefore the greenish

function we defined there and more generally above is

related to G of 2/D prob by $g(\vec{x}, \vec{y}) = \hat{n} \cdot \nabla_y G$

$$u(\theta) = \int_0^{2\pi} d\theta' \frac{1}{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta - \theta')} f(\theta')$$

$$\begin{array}{lll} D=3 & \vec{x} \in V & \vec{y} \in \partial V \\ D=2 & \vec{x} \in R & \vec{y} \in C = \partial R \end{array}$$

The symmetry prop $G(\vec{x}, \vec{y}) = G(\vec{y}, \vec{x})$ was a property of the spectral rep of G . Use Green's formula to prove it in a more fundamental way from Green 2.

$$\int_V d^2x [u(x) \nabla^2 v(x) - v(x) \nabla^2 u(x)] = \int_{\partial V} d^2x [u(x) \mathbf{n} \cdot \nabla v(x) - v(x) \mathbf{n} \cdot \nabla u(x)]$$

$\downarrow = 0$ $\downarrow = 0$

Take $u(x) = G(\vec{x}, \vec{y})$ $v(x) = G(\vec{x}, \vec{y}')$

$$\int d^2x [G(\vec{x}, \vec{y}) \delta(\vec{x} - \vec{y}') - G(\vec{x}, \vec{y}') \delta(\vec{x} - \vec{y})] = 0$$

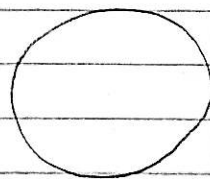
$$G(\vec{y}', \vec{y}) - G(\vec{y}, \vec{y}') = 0$$

Poisson's eqn in disc of radius a .

$$\nabla^2 u = \rho(r) \quad |x| \leq a$$

$$u(r=a, \theta) = 0$$

Compare Sec 7.7 of H



General thy suggests that there is a $G(\vec{x}, \vec{y})$ which has spectral rep

$$G(\vec{x}, \vec{y}) = \sum \frac{\phi_n(\vec{x}) \phi_n(\vec{y})}{\lambda_n N_n^2}$$

We would like to find the eigenfunc $\phi_n(r)$ and eigenvalues λ_n which satisfy $\nabla^2 \phi = -\lambda \phi$

We can use much of our previous work on $\mathcal{L}/\mathcal{D}$ prob in pp 67-69

1. Because of circ. sym. it is best to use polar coords r, θ in which $\nabla^2 \phi(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial \phi}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = -\lambda \phi(r, \theta)$

2. By sep. of vars. there are product solns

$$\phi(r, \theta) = \phi_{nm}(r) e^{im\theta} \quad -\infty \leq m \leq \infty$$

Subst in PDE and find that radial eigenfunc. satisfy ODE

$$\phi_{nm}'' + \frac{1}{r} \phi_{nm}' - \frac{m^2}{r^2} \phi_{nm} = -\lambda_{nm} \phi_{nm}$$

and BC: $\phi_{nm}(r=a) = 0$ $\lim_{r \rightarrow 0} \phi_{nm}(r)$ exists

The difference between eigenval. prob + $\mathcal{L}/\mathcal{D}$ problem is that the λ_{nm} appear on r.h.s., and we should expect to find an ∞ # of λ 's.

Note that this is a 2-dim problem as was the reg. prob on $\square$ in p 60-61.

We found $\phi_{nm}(x,y) = \sin nx \sin ny$ $\lambda_{nm} = m^2 + n^2$

We clearly needed 2 labels n, m to label an eigenmode uniquely, and λ_{nm} depends on both labels.

Same is true for our present prob on $\bigcirc$. The integer m labels the angular mode, but the value of m enters the radial eqn. We expect an ∞ # of radial eigenfns for each value of m , so we have labeled $\phi_{nm}(r)$. We expect λ_{nm} to depend on both n and m .

Further, previous analysis using Green's formula tells us that $\lambda_{nm} > 0$ for this Dirichlet prob.

3. The radial ODE can be transformed to Bessel's Eq of order m by change of variable $z = \sqrt{\lambda} r$. See p. 314 of H
Sols which are reg at origin are:

$$\phi_{nm}(r) = J_m(\sqrt{\lambda_{nm}} r)$$

4. The final reg. is to impose BC $\phi_{nm}(a) = J_m(\sqrt{\lambda_{nm}} a) = 0$
Our early work on Bessel fns and Sec 7.7 of H $\Rightarrow$

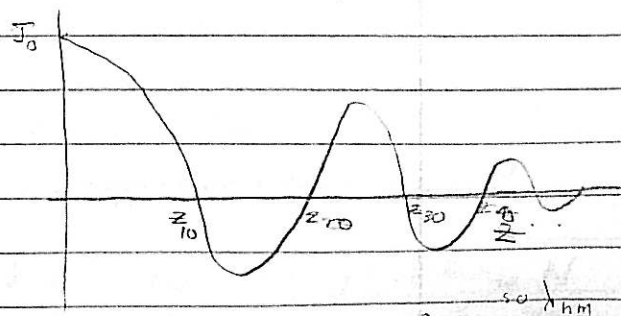
The $J_m(z)$ has an ∞ # of zeros for $0 < z < \infty$ and decays in magnitude with $1/\sqrt{z}$ envelope

is Evid H

The eigenvalues are selected by "fitting these waves into disc"

Specifically we choose

$$\sqrt{\lambda_{nm}} a = z_{nm}$$



5. Spectral rep. Use $J_m(z) = -J_{-m}(z) = J_{|m|}(z)$ so $\lambda_{nm} = \lambda_{n-m}$

$$G(r, \theta; r', \theta') = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} J_m(\sqrt{\lambda_{nm}} r) J_m(\sqrt{\lambda_{nm}} r') e^{im\theta} e^{-im\theta'}$$

Ex: combine m and $-m$ terms to get

$$G(r, \theta; r', \theta') = \sum_{n=1}^{\infty} \frac{J_0(\sqrt{\lambda_{n0}} r) J_0(\sqrt{\lambda_{n0}} r')}{-\lambda_{n0} N_{n0}^2} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{J_m(\sqrt{\lambda_{nm}} r) J_m(\sqrt{\lambda_{nm}} r')}{-\lambda_{nm} N_{nm}^2} \cos m(\theta - \theta')$$

Correct but ugly and not ^{very} useful formula:

One thing that is guaranteed is that the Green's fn

$$g(r, \theta - \theta') = \frac{1}{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta - \theta')} = \left. \frac{\partial}{\partial r'} G(r, \theta; r', \theta') \right|_{r'=a}$$

Later in the term, we will obtain a rather simple closed expression for G using method of images.

Illustration of the completeness relation for eigenfn of 1d case;

$$\frac{d^2}{dx^2} \phi = -\lambda \phi \quad -\pi < x < \pi \quad \text{Periodic BC's:}$$

We know that $\phi_n(x) = e^{cnx} \quad -\infty < n < \infty$

The eigenfn expansion is just the full F. series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{cnx} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dy e^{-cny} f(y).$$

The formal statement of completeness is

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{cn(x-y)} = \delta(x-y)$$

Truncate mode sum at $\pm N$ to define Dirichlet kernel

$$\Delta_N(x-y) = \frac{1}{2\pi} \sum_{n=-N}^N e^{cn(x-y)}$$

Outline of our strategy:

- 1) Obtain simple closed expression for Δ_N
- 2) Convince you that it is a δ -approximant as $N \rightarrow \infty$

Hence Δ_N looks, feels, smells and acts like $\delta(x-y)$ at large N

Since the completeness relation was obtained from assumption that F. series is valid, our discussion will really be a good outline of rigorous pf of F. series

Historical note:

1) Claim $\Delta_N(x-y) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})(x-y)}{\sin\frac{1}{2}(x-y)}$

can set $y=0$ to simplify d is common.

Pf: consider

$$\begin{aligned} & \left(e^{i\frac{y}{2}} - e^{-i\frac{y}{2}} \right) \sum_{n=-N}^N e^{inx} = \\ & e^{i(N+\frac{1}{2})x} + e^{i(N-\frac{1}{2})x} + e^{i(N-\frac{3}{2})x} + \dots + e^{i(-N+\frac{1}{2})x} \\ & - e^{i(N-\frac{1}{2})x} - e^{i(N-\frac{3}{2})x} - \dots - e^{i(-N-\frac{1}{2})x} \\ & = e^{i(N+\frac{1}{2})x} - e^{-i(N-\frac{1}{2})x} \quad \text{due to almost complete cancellation.} \end{aligned}$$

Hence $\Delta_N(x) = \frac{1}{2\pi} \sum_{n=-N}^N e^{inx} = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}$

Properties of $\Delta_N(x)$

i) $\int_{-\pi}^{\pi} \Delta_N(x) dx = \frac{1}{2\pi} \sum_{n=-N}^N \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi} \cdot 2\pi = 1$

← only $n=0$ is non zero

ii) $\lim_{x \rightarrow 0} \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x} = \frac{1}{2\pi} (2N+1)$ by L'Hopital's rule

iii) $\lim_{x \rightarrow \pm\pi} \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x} = \frac{1}{2\pi} (-1)^N$

iv) $\Delta_N(x)$ has $2N$ internal zeros at $(N+\frac{1}{2})x_k = k\pi$

$$x_k = \frac{k\pi}{N+\frac{1}{2}} \quad -N < k < N \quad k \neq 0$$

Spacing $\Delta x = \frac{\pi}{N+\frac{1}{2}}$

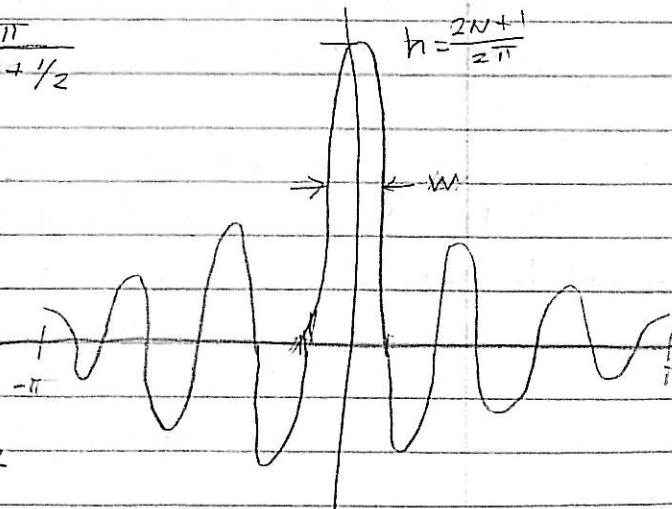
Plot (for large even N)

Features: a) high narrow central peak

height $h = \frac{2N+1}{2\pi}$

width $w = \frac{2\pi}{N+\frac{1}{2}}$

b) $2N$ zeros spaced by $\Delta x = \frac{\pi}{N+\frac{1}{2}}$



For a smooth $f(x)$, consider

$$I_N = \int_{-\pi}^{\pi} dx \Delta_N(x) f(x) \xrightarrow[N \rightarrow \infty]{?} f(0)$$



What can we say about limit of subintegral⁻ (for any $0 < a < \pi$)

$$\lim_{N \rightarrow \infty} \int_a^\pi dx \Delta_N(x) f(x) = \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_a^\pi dx \sin(N+\frac{1}{2}x) \frac{f(x)}{\sin \frac{1}{2}x}$$

$= 0$

$\left[\begin{array}{l} \text{ratio is} \\ \text{contin. in} \\ [a, \pi] \end{array} \right]$

Limit vanishes by Riemann-Lebesgue Lemma!

$$\text{So } \lim_{N \rightarrow \infty} I_N = \lim_{N \rightarrow \infty} \int_{-\varepsilon}^{\varepsilon} dx \Delta_N(x) f(x) = f(0)$$

Choose $\varepsilon \rightarrow 0$ small enough so that only the central peak contributes to limit.

It takes an 18,100 type argument to show that.

Here is a heuristic version of that arg.

i) take ε so small that we can accurately approx $f(x) \approx f(0)$

$$\text{Have } \lim_{N \rightarrow \infty} \int_{-\varepsilon}^{\varepsilon} dx \Delta_N(x) f(x) = f(0) \lim_{N \rightarrow \infty} \int_{-\varepsilon}^{\varepsilon} dx \Delta_N(x)$$

ii) In limit of large N we can add back the regions that contribute zero by R-L lemma, so

$$\lim_{N \rightarrow \infty} \int_{-\varepsilon}^{\varepsilon} dx \Delta_N(x) = \lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} dx \Delta_N(x) = 1.$$

Then (long) argument shows that $\Delta_N(x-y)$ behaves like $\delta(x-y)$ at large N . In this way we can understand the completeness relation $\sum_n \frac{\phi_n(x) \phi_n^*(y)}{N_n^2} = \delta(x-y)$

New Topic: Heat eq on ∞ line and Sol. by Fourier Exp: ch 10 of H

statement of prob: $\partial_t^2 u(x,t) = \partial_x^2 u(x,t)$ on $\mathbb{R} = (-\infty, \infty)$

Previous BC's $\rightarrow$ growth cond. $u(x,t)$ bounded at large $|x|$
(needed for unique sol.)

Imitate our past successes with sep. of vars.

$$u(x,t) = T(t) \phi(x)$$

$$\Rightarrow \text{ODEs } \dot{T}(t) = -\lambda T(t) \quad \phi''(x) = -\lambda \phi(x)$$

Consider sets of $\phi(x)$ for various values of λ :

$$i) \lambda = -\mu^2 < 0 \quad \phi_{\text{gen}}(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

both violate our growth cond and can be discarded