

18.303 Lecture 1 10 Sept 2009

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PDE's (and ODE's) describe a vast range of scientific phenomena. In 18.303 we want to learn

- what the basic eqns are
 - techniques to find sols., both exact and approx.
 - general insights into wave motion, heat flow, etc
- These are the physical phenomena explained by PDE's

Today's Plan; 1) Brief examples to illustrate ODE's vs PDE's

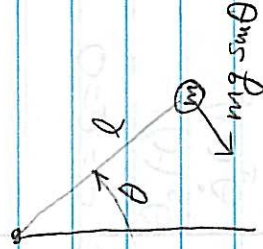
- Mechanics of the course
- more on PDE's

18.03
never used

I. ODE's describe fns. depending on 1 indep. variable

They arise in many ways, e.g.

Newton's law for the motion of pendulum



$$m l \frac{d^2 \theta}{dt^2} = -mg \sin \theta$$

$$\ddot{\theta} = -\omega^2 \sin \theta \quad \omega = \sqrt{\frac{g}{l}}$$

$$\text{Ex! } [\omega] = 1/\text{sec}$$

Initial Value Prob. (IVP)

Find sol. $\theta(t)$ which satisfies IC's $\theta(t=0) = \theta_0$

$$\dot{\theta}(t=0) = \omega_0$$

$\exists$ unique sol to this prob. $\iff \exists$ a
general sol which depends on two arbit. const's.

$$\theta_{\text{gen}}(t) = \theta(t, \theta_0, \omega_0)$$

A difficulty, known for >100 years, sols. exist, but they
can't be expressed in terms of elem. fns

e.g. t^n , e^{at} , $\sin \omega t$ etc.

(Instead, study of the ODE leads to a new class of
special fns, called elliptic fns in this case.)

Much of the diffc. occurs because the ODE is nonlinear
Dependent var., occurs non-linearly in $\sin \theta$

One can proceed by physical + mathem. arguments to find a linear ODE whose sols. approximate the non-linear ones under reasonable conds.

a) physically, we know if we are initially at small θ_0 and small ω_0 , then $\theta(t)$ and $\dot{\theta}(t)$ remain small

Conds. $\theta_0 \ll 1$ (radians) $\omega_0 \ll \omega = \sqrt{g/L}$

b) Mathematically, we put series $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$

For small θ the approx $\sin \theta \approx \theta$ is valid.

So we replace the difficult nonlinear ODE by an approx. linear one valid for small angles.

$$\ddot{\theta} = -\omega^2 \theta$$

Then a linear $\exists$ a pretty method. structure associated with it. Best of all, we can solve it exactly.

Let's discuss those things in a review of basic 18.03.

c) linearity of $\theta_1(t)$ and $\theta_2(t)$ are any two sol.,

then $c_1 \theta_1(t) + c_2 \theta_2(t)$ is also a sol.

(i) two sols are lin indep if $c_1 \theta_1(t) + c_2 \theta_2(t) \equiv 0 \Rightarrow c_1 = c_2 = 0$

$$\text{Alternate Wronskian criterion } W(\theta_1, \theta_2) \equiv \begin{vmatrix} \theta_1(t) & \theta_2(t) \\ \dot{\theta}_1(t) & \dot{\theta}_2(t) \end{vmatrix}$$

Linear indep $\Leftrightarrow W(\theta_1, \theta_2) \neq 0$ at any fixed t .

(ii) Gen sol. is a superposition of any two lin. indep sols

$$\theta_{\text{gen}}(t) = c_1 \theta_1(t) + c_2 \theta_2(t)$$

iv) Can choose $\theta_1(t) = \cos \omega t$ $\theta_2(t) = \sin \omega t$

Sol. of IVP is $\theta(t) = \theta_0 \cos \omega t + \frac{\omega_0}{\omega} \sin \omega t$

If $\theta_0 \ll 1$ and $\omega_0 \ll \omega$, then is a good approx. to motion of the nonlinear pendulum.

Important: Properties i)-iii) hold for general 2nd order

linear ODE's $y'' + p(t)y' + q(t)y = 0$

if $q(t), p(t)$ satisfy certain smoothness properties (to be discussed later)

The general thm. guarantees the existence of 2 lin indep

sols $y_1(t), y_2(t)$ but we cannot always find them in closed form.

II PDE's determine fns of 2 or more indep. variables.
typically one time t and one or more space coords x, y

(Spirit of our discussion of PDE's is opposite to what was just said about ODE's. The latter was a review of 18.03. What I say now about PDE's will be derived later in a more thorough way. So don't expect to understand it all now.)

Basic linear PDE: heat eq. or diffusion eq.

Physical set up: x, y plane covered by thin layer of H_2O

Insert drop of ink at time $t=0$ at pt (x_0, y_0) .

We want to learn how it will diffuse.

Let $u(x, y, t)$ denote density of ink at time t at pt (x, y) .

Later we will derive that it satisfies the 2-dim diffusion eq.

$$\frac{\partial}{\partial t} u(x, y, t) = k \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y, t)$$

$$\partial_t u = k (\partial_x^2 + \partial_y^2) u$$

Properties: i) 1st order in t , 2nd order in x, y .

ii) linear since u appears to first order only.

⇒ Sup. Principle if $u_1(x, y, t)$ and $u_2(x, y, t)$ are any two sol's, then $c_1 u_1(x, y, t) + c_2 u_2(x, y, t)$ is also a sol.

Ex: Prove this by substit. in the PDE, as in 18.03.

Idea of derivation a) randomness diffusion results from random collision of ink 'molecules' and water molecules.

k is diffusion constant $[k] = l^2/t$ eg. cm^2/sec

b) For heat flows one can derive heat eqn from conservation of thermal energy. See Haberman Ch 1

Applications: i) heat flow ii) diffusion of pollutants

iii) mathematical finance (Black-Scholes)

First look at mathematics of heat eqn. - study a 1-dim version. Diffusion or heat flow on line.

Problem find $u(x,t)$ such that $\partial_t u = k \partial_x^2 u$.

Look for special class of separable sols. (a product sol.)

$$u(x,t) = T(t) \phi(x)$$

$$u_t = \frac{\partial}{\partial t} \quad , \quad = \frac{\partial}{\partial x}$$

Subst in PDE: $\dot{T}(t) \phi(x) = k T(t) \phi''(x)$

assume intervals in t, x in which $T(t) \phi(x) \neq 0$.

divide

$$\frac{\dot{T}(t)}{k T(t)} = \frac{\phi''(x)}{\phi(x)}$$

LHS depends on t only, RHS depends on x only. $\Rightarrow$

$$\frac{\dot{T}(t)}{k T(t)} = -\lambda = \frac{T(t)}{k T(t)} \quad \lambda \text{ is separation const.}$$

PDE $\rightarrow$ two ODE's which "share" value of λ

i) $\phi''(x) = -\lambda \phi(x)$ ii) $\dot{T}(t) = -k\lambda T(t)$

Assume λ real > 0 . Justify this later.

i) is an eigenvalue prob. for differential op. $D = \frac{d}{dx}$.

$$D^2 \phi_\lambda(x) = -\lambda \phi_\lambda(x) \quad \text{Vocab, } \lambda \text{ is an eigenvalue}$$

$\phi_\lambda(x)$ is an eigenfn.

(Same "structure" on 2×2 matrix eigenv. prob of 18.03

$$A \vec{v} = \lambda \vec{v})$$

Simple fact for each $\lambda > 0$, there are two lin indep. sols.

$$\phi_\lambda^{(1)} = \sin(\sqrt{\lambda} x) \quad \phi_\lambda^{(2)} = \cos(\sqrt{\lambda} x)$$

ii) Sol of $T(t)$ ODE is $T_\lambda(t) = e^{-k\lambda t}$

Put things together and write: $u_\lambda(x,t) = e^{-k\lambda t} (b(\lambda) \sin \sqrt{\lambda} x + a(\lambda) \cos \sqrt{\lambda} x)$

$\Rightarrow$ For each $\lambda > 0$, $\exists$ two lin indep sols, but altogether $\exists$ ∞ set of sols. (very different from 2nd order ODE with 2 and only 2 indep sols.)

Important fact (to be developed) Gen sol of PDE can be

expressed as a superp. of an ∞ number of separable sols. So the sep. sols. are very basic and very useful.

Digression on ~~our~~ structure of course:

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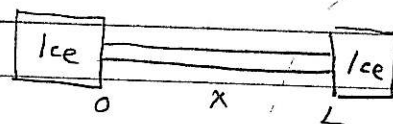
1. discuss outline comment no numerical methods for PDE's.
2. TA + off hr. TBA "open door policy"
3. student feedback important - ~~PDE~~ "18.303 was useful but dull"
4. email contact encouraged
5. Web site <http://math.mit.edu/classes/18.303/>

Typical PDE problem on finite interval $0 \leq x \leq L$

3diments: PDE $\partial_t u = k \partial_x^2 u$ $0 \leq t$ $0 \leq x \leq L$

Bdy. conds. BC(s) $u(0, t) = 0$

$$u(L, t) = 0$$



Init cond. IC $u(x, 0) = f(x)$

The physical prob. is to find temp dist. $u(x, t)$ in a uniform rod in thermal contact with 'heat baths' at 0°C .

Goal: Find sol of PDE which satisfies BC's and IC.

This problem has a unique sol, as we will show later.

Our purpose today is to outline how to construct the sol.

using the sep sols $T_\lambda(t) \phi_\lambda(x)$.

Step 1: impose BC's on previous eigenfns

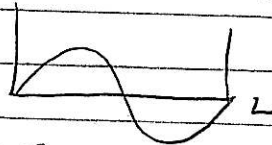
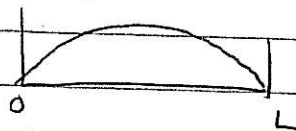
a) $\phi_\lambda(0) = 0$ $\sin \sqrt{\lambda} x$ satisfies then $\forall \lambda$
 $\cos \sqrt{\lambda} x$ does not, so we discard the

'cosine modes' and keep the 'sine modes', $\phi_\lambda(x) = \sin \sqrt{\lambda} x$

b) $\phi_\lambda(L) = 0$ we must choose the eigenvalue λ so that the "wave" "fit" into the physical interval.

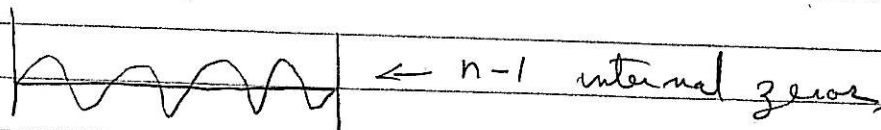
$$\sqrt{\lambda} = \frac{\pi}{L} \quad \sin \frac{\pi}{L} x$$

$$\sqrt{\lambda} = \frac{2\pi}{L} \quad \sin \frac{2\pi}{L} x$$



In general we find allowed

eigenvals $\sqrt{\lambda}_n = \frac{n\pi}{L}$ and eigenfns. $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$



So we have a large set of sep. solr $u_n(x,t) = e^{-\frac{n^2\pi^2}{L^2}t} \sin \frac{n\pi x}{L}$
 Each u_n satisfies PDE and BC, but not IC $u(x,0) = f(x)$

Step 2: $\exists$ unique superp. of the $u_n(x,t)$ which satisfies full
 Finding this sol. will lead us to Fourier series prob.
 as in 18.03 but let's proceed from viewpoint of PDE.

Two ideas (i) superpos. principle (as version)

Since PDE is linear $b_1 u_1(x,t) + b_2 u_2(x,t)$ is a sol.
pf by subst.

$$\partial_t(\quad) = k \partial_x^2(\quad)$$

$$b_1 \cancel{\partial_t} u_1 + b_2 \cancel{\partial_t} u_2 = b_1 k \cancel{\partial_x^2} u_1 + b_2 k \cancel{\partial_x^2} u_2 \text{ done}$$

Any sup. $b_1 u_1 + b_2 u_2 + b_3 u_3$ is also a sol

by same pf.

One might then believe that an ∞ sup. of sols is a sol.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n^2\pi^2}{L^2}t\right)}$$

Note very rapid fall off in n of $L^2 e^{-n^2\pi^2 t}$ for any

This should reassure us that the ∞ $t > 0$
 series is meaningful and is still a soln.

$$\lim_{t \rightarrow 0} u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \stackrel{?}{=} f(x)$$

(ii) idea of eigenfn expansion in this case Fourier series

Fact from 18.03: a) if the initial temp dist. is

piecewise smooth, then $f(x)$ can be represented by its

F. sine series. $\lim_{N \rightarrow \infty} \sum_{n=1}^N b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f(x) & \text{if } f(x) \text{ is contin. at } x \\ \frac{1}{2}(f(x_+) + f(x_-)) & \text{at } x \end{cases}$

b) the F. coeff. are given by

$$b_n = \frac{2}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) f(x)$$

"jump" of f

c. The coeffs b_n have the property $b_n \rightarrow 0$ as $n \rightarrow \infty$ 7
 at least as fast as $1/n$ (explains convergence at $t=0$)

d) The eigenfns are orthogonal on the interval $[0, L]$

$$\int_0^L dx \phi_n(x) \phi_m(x) = 0 \text{ if } n \neq m$$

These basic properties of F. series are discussed in 18.03.

Please review this topic. It will be important for us.

The orthog. relation * can be proven in several ways.

Let's do it by an interesting method which will apply to many other situations in 18.303

1. Since $\lambda_n \neq \lambda_m$ we study $I_{mn} \equiv (\lambda_n - \lambda_m) \int_0^L dx \phi_n(x) \phi_m(x)$

Recall eigen v. prob $\phi'' = -\lambda \phi$, so

$$I_{mn} = - \int_0^L dx (\phi_m''(x) \phi_n(x) - \phi_m(x) \phi_n''(x))$$

$$= - \int_0^L dx \frac{d}{dx} (\phi_m'(x) \phi_n(x) - \phi_m(x) \phi_n'(x))$$

$$= \text{use } \int_a^b dx \frac{d}{dx} f(x) = f(b) - f(a)$$

$$= - (\phi_m' \phi_n - \phi_m \phi_n') \Big|_0^L = 0 \text{ by our BC's } \phi_n(x) = 0 \text{ at } x=0, L$$

Pf. is complete.

Tues Sept 15 Summary of 1st lecture:

1. 2nd order linear ODE's e.g. $\ddot{\theta} + \omega^2 \theta = 0$

$\exists$ 2 lin indep sol. $\theta_1(t) = \cos \omega t$ & $\theta_2(t) = \sin \omega t$

Gen. sol. is superp. $\theta_{\text{gen}}(t) = c_1 \theta_1(t) + c_2 \theta_2(t)$

2. Linear PDE's such as Heat Eq on line Ref ch 2 of Haberman

$$\partial_t u(x,t) = k \partial_x^2 u(x,t)$$

$\exists$ ∞ set of separable sol. $u_\lambda(x,t) = (b(\lambda) \sin \sqrt{\lambda} x + a(\lambda) \cos \sqrt{\lambda} x) e^{-k\lambda t}$

Finite interval $0 \leq x \leq L$ with

Bdy. cond's such as $u(0,t) = 0$ $u(L,t) = 0$

(BC's correspond to physical cond's on system)

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

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BC's select a discrete set of sep. sol's:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} kt}$$

$e^{-n^2 a} \xrightarrow[n \rightarrow \infty]{\text{Super fast}}$

Any sol. can be expressed as superp.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2 \pi^2}{L^2} kt}$$

← Superfast convergence for any $t > 0$

∃ unique sol which satisfies PDE, BC's and I.C.

$$u(x,0) = f(x)$$

Soln given by F. sine series (special case of eigenfn. expansion).

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) f(x)$$

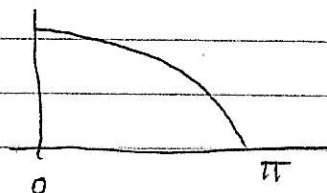
Example: heat eqn on $\boxed{0 \leq x \leq \pi}$

$$\partial_t u = \partial_x^2 u \quad \text{on } 0 \leq x \leq \pi$$

choose $k=1$ (which means a choice of units of time.)

Ice BC's and

$$\text{I.C. } u(x,0) = \cos\left(\frac{x}{2}\right)$$



The F. coeff are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} dx \sin nx \cos \frac{x}{2}$$

Use inverse Euler formula $\sin nx = \frac{1}{2i} (e^{inx} - e^{-inx})$

$$\cos \frac{1}{2} x = \frac{1}{2} (e^{ix/2} + e^{-ix/2})$$

$$b_n = \frac{2}{\pi} \frac{1}{4i} \int_0^{\pi} dx (e^{inx} - e^{-inx}) (e^{ix/2} + e^{-ix/2})$$

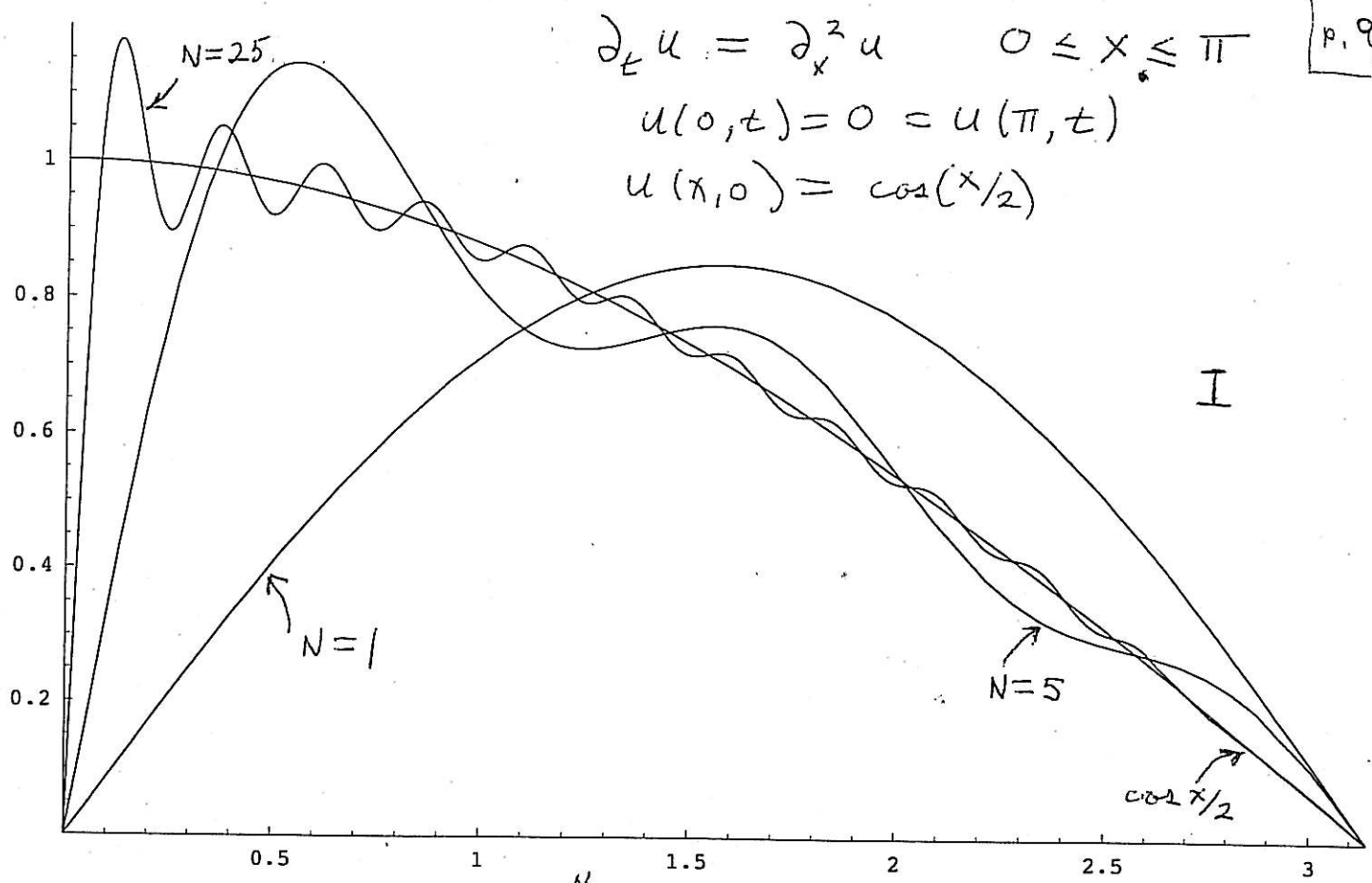
$$\Rightarrow (e^{i(n+\frac{1}{2})x} + e^{i(n-\frac{1}{2})x} - e^{-i(n-\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x})$$

$$= \frac{1}{\pi} \int_0^{\pi} dx [\sin(n+\frac{1}{2})x + \sin(n-\frac{1}{2})x]$$

$$= -\frac{1}{\pi} \left[\frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} + \frac{\cos(n-\frac{1}{2})x}{n-\frac{1}{2}} \right] \Big|_0^{\pi}$$

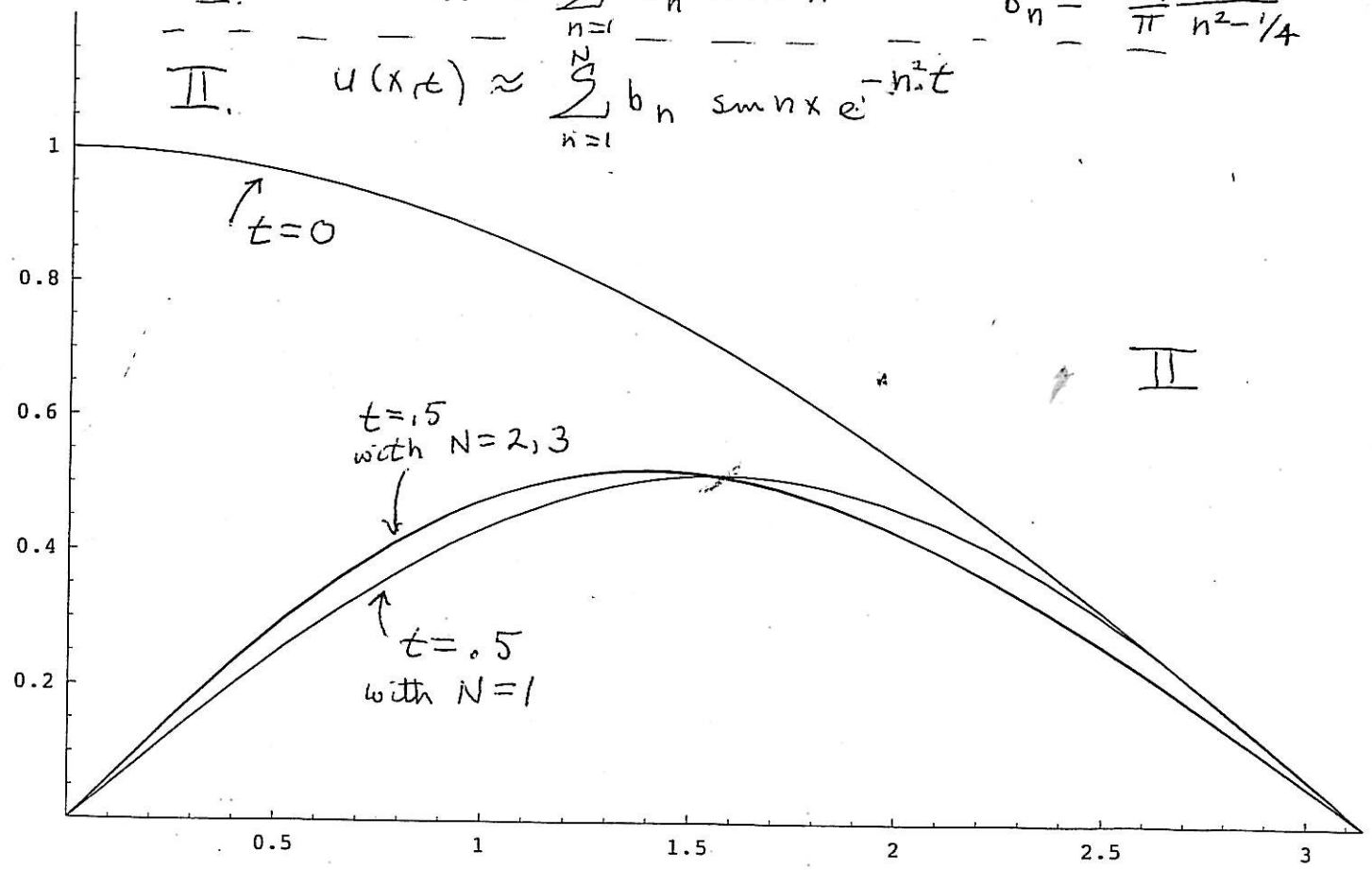
$$= \frac{1}{\pi} \left[\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}} \right] = \frac{2}{\pi} \frac{n}{n^2 - 1/4} \xrightarrow[n \rightarrow \infty]{} 0 \text{ as } 1/n$$

Expect convergence of series



I. $\cos x/2 \approx \sum_{n=1}^N b_n \sin nx$ $b_n = \frac{2n}{\pi} \frac{1}{n^2 - 1/4}$

II. $u(x, t) \approx \sum_{n=1}^N b_n \sin nx e^{-n^2 t}$



Graph of results (from MAPLE or Mathematica)

I. series for initial data $\cos \frac{x}{2}$ very slow convergence with indices of Gibbs phenomenon near $x=0$

II. series for $u(x,t) \approx \sum_{n=1}^N b_n \sin(n, x) e^{-n^2 t}$ for $t \leq .5$

Ultra-fast conv., including $n=2$ mode gives small correction to $N=1$ approx. Adding $n=3$ mode has no visible effect.

Large t behavior $\lim_{t \rightarrow \infty} u(x,t) = \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} b_n \sin n x e^{-n^2 t} \equiv 0$

All parts of the bar come to equilibrium with ice at end ptr.

Two core ideas for linear PDE's:

1. Gen sol is superp. of separable sols which come from bdy value prob. or eigenvalue prob. (We have now illustrated this)

2. Gen sol can be expressed using Green's fn

$$u(x,t) = \int_0^{\pi} dy G(x-y,t) f(y) \quad \text{for any initial data}$$

One of the main goals of 18.303 is to understand Green's fns.

(Several lectures on) Properties of 2nd order linear ODE's. (important for us because ODE's occur when we apply sep. of variables to PDE's.)

$$\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t)y = 0 \iff \ddot{y} + P(t) \dot{y} + Q(t)y = R(t)$$

homog. 2nd order linear alternate notation inhom. source

Regularity conditions are req'd to make the thg we develop valid.

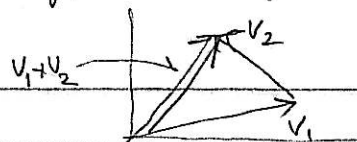
Assume $P(t)$, $Q(t)$, $R(t)$ are continuous on $[a, b]$.

Some facts

1. The homog. ODE has 2 linearly indep sols $y_1(t)$ and $y_2(t)$ (which can be chosen in many ways.)

2. Gen sol is superp. $y_{\text{gen}}(t) = c_1 y_1(t) + c_2 y_2(t)$

Think of "space of sol" as a
2-dim. vector space



(We will develop this idea soon).

3. Wronskian gives a practical test of lin. indep. (and is useful for several other things).

Given two sol, $y_1(t)$, $y_2(t)$, then Wronskian is

$$W(y_1, y_2) = y_1 \dot{y}_2 - y_2 \dot{y}_1$$

if $W(y_1, y_2) \neq 0$ on $[a, b]$, then y_1, y_2 are indep. (and conversely)

It is sufficient to check this at any single pt. of $[a, b]$

Lemma: either $W(y_1, y_2)$ vanishes everywhere on $[a, b]$ or not at all

Pf. derive and solve a simple ODE for W .

$$\begin{aligned} \frac{d}{dt} W(t) &= \dot{y}_1 \dot{y}_2 + y_1 \ddot{y}_2 - \dot{y}_2 \dot{y}_1 - y_2 \ddot{y}_1 && \text{use homog ODE} \\ &= -P(t)(y_1 \dot{y}_2 - y_2 \dot{y}_1) - Q(t)(y_1 \dot{y}_2 - y_2 \dot{y}_1) \\ &= -P(t) W(t) \end{aligned}$$

Wronskian obeys 1st order separable ODE

$$\frac{1}{W} \frac{dW}{dt} = -P(t)$$

sol $W(t) = C e^{-\int P(t) dt}$ ← exponential cannot vanish if $P(t)$ is contin, so either

i) $C \neq 0 \Rightarrow W(t)$ does not vanish in $[a, b]$, or

ii) $C = 0 \Rightarrow W(t)$ vanishes everywhere.

Assignment: review 18.03 method for const coeff ODE's

$$\ddot{y} + p \dot{y} + q y = 0$$

Method: $y = e^{rt} \rightarrow r^2 + pr + q = 0$

1) $p^2 - 4q > 0$ real exp. sols. $e^{r_1 t}$ $e^{r_2 t}$
2) $p^2 - 4q < 0$ cx. exp. sols. $y(t) = e^{-p/2 t} e^{i \omega t}$, $y^*(t)$
2) $p^2 - 4q = 0$ $y_1 = e^{-p/2 t}$ $y_2 = t e^{-p/2 t}$

Reduction of Order Method: a systematic way to find $y_2(t)$, if given $y_1(t)$

1) Suppose $y_1(t)$ is a sol of $\ddot{y} + P(t)y + Q(t)y = 0$

2. Assume $y_2(t) = v(t)y_1(t)$

$$\dot{y}_2 = v\dot{y}_1 + \dot{v}y_1$$

$$\ddot{y}_2 = v\ddot{y}_1 + 2\dot{v}\dot{y}_1 + \ddot{v}y_1$$

3. Subst. in ODE

$$v\cancel{\ddot{y}_1} + 2\dot{v}\dot{y}_1 + \ddot{v}y_1 + P(v\cancel{\dot{y}_1} + \dot{v}y_1) + Qv\cancel{y_1} = 0$$

We are left with $\ddot{v}y_1 + (2\dot{y}_1 + Py_1)\dot{v} = 0$

4. Separ. eqn for $\dot{v}$:

$$\frac{\ddot{v}}{\dot{v}} = -2\frac{\dot{y}_1}{y_1} - P$$

$$\ln \dot{v} = -\ln y_1^2 - P$$

$$\dot{v} = \frac{1}{y_1^2} e^{-\int^t P(t) dt}$$

5. One more integration to find

$$v(t) = \int^t dt \frac{1}{y_1^2(t)} e^{-\int^t P(t) dt}$$

$$y_2(t) = v(t)y_1(t)$$

"Remember the method, not the formula" $n > 0$ integer

Apply to Legendre eqn: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

This ODE arises when sep. of vars. is applied to linear PDE's in 3 dim in spherical coords. (Example Schrödinger eq for wave fun of hydrogen).

Observe For every +ve integer n , one sol is an n th order poly.

$$n=0 \quad P_0(x) = 1 \quad n=1 \quad P_1(x) = x \quad n=2 \quad P_2(x) = 3x^2 - 1$$

For $n=1$, it is easy to check that $y_1(x) = x$ is a sol.

Use R of O method to find y_2

Assume $y_2 = v x$ $y_2' = v + x v'$ $y_2'' = x v'' + 2v'$

Subst in $(1-x^2)y'' - 2xy' + 2y = 0$

$$(1-x^2)(xv'' + 2v') - 2x(v + xv') + 2v = 0$$

$$(1-x^2)xv'' + 2(1-x^2-x^2)v' = 0$$

$$\frac{v''}{v'} = \frac{-2(1-x^2-x^2)}{(1-x^2)x} = -\frac{2}{x} + \frac{2x}{1-x^2}$$

$$\ln v' = -2 \ln x + \int \frac{2x}{1-x^2} dx = -2 \ln x - \ln(1-x^2)$$

$$\Rightarrow v' = \frac{1}{x^2(1-x^2)} \Rightarrow v(x) = -\frac{1}{x} + \frac{1}{2} \ln \frac{x+1}{x-1}$$

Can use partial fractions on MAPLE.

So $y_2(x) = v y_1 = -1 + \frac{1}{2} x \ln \frac{x+1}{x-1} = Q_1(x)$

Std notation for Legendre fn of 2nd kind $\uparrow$ (which occurs in many practical applications.)

Method of variation of parameters for:

$$\ddot{y} + P(t)\dot{y} + Q(t)y = R(t) \quad \leftarrow \text{source}$$

Gen sol $y_{\text{gen}} = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

MVP method constructs $y_p(t)$ given y_1, y_2 $\left\{ \begin{array}{l} \text{any partic. sol} \\ \text{of inhom. ODE} \end{array} \right.$

Assume $y(t) = y_p(t) = v_1 y_1 + v_2 y_2$

strategy: sub in ODE which will give one eq relating v_1, v_2 . Impose one further relation betw v_1, v_2 which will allow us to solve the system.

Let's do it: (i) compute $\dot{y} = v_1 \dot{y}_1 + v_2 \dot{y}_2 + \dot{v}_1 y_1 + \dot{v}_2 y_2$

(ii) Impose $\dot{v}_1 y_1 + \dot{v}_2 y_2 = 0$ Motivation: avoid $\dot{v}_1$ and $\dot{v}_2$ which would appear in next step.

$$\ddot{y} = v_1 \ddot{y}_1 + v_2 \ddot{y}_2 + \dot{v}_1 \dot{y}_1 + \dot{v}_2 \dot{y}_2 + P(v_1 \dot{y}_1 + v_2 \dot{y}_2) + Q(v_1 y_1 + v_2 y_2) = R(t)$$

We are left with: $\dot{v}_1 y_1 + \dot{v}_2 y_2 = R(t)$

and our cond. $\dot{v}_1 y_1 + \dot{v}_2 y_2 = 0$