

$$= \frac{2}{k|x|} \sin k|\vec{x}|$$

Put things together

$$G(x) = -\frac{1}{(2\pi)^3} (2\pi) \cdot \frac{2}{|x|} \int_0^\infty \frac{dk}{k} \sin k|x|$$

and then

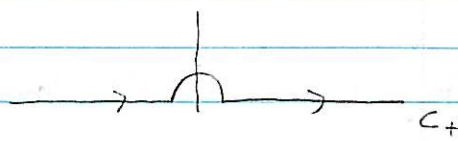
$$\equiv -\frac{1}{2\pi^2|x|} \int_0^\infty \frac{dv}{v} \sin v \quad v = k|x|$$

The remaining integral is simply a constant^H, but let's evaluate because it is a nice example of contour methods

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dv \frac{\sin v}{v} \quad \text{true since } f(v) = \frac{\sin v}{v} \text{ is even}$$

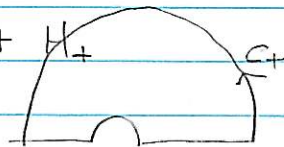
↑ This $f(v)$ has no poles anywhere ("pole" at $v=0$ cancels)

= Cauchy's Thm allows to distort the contour

$$H = \frac{1}{2} \int_{C_+} dv \frac{\sin v}{v}$$


The rigorous reason is that $\int_{-\infty}^{+\infty} - \int_{C_+} = \int = 0$

$$H = \frac{1}{4i} \int_{C_+} dv \left[\frac{e^{iv}}{v} - \frac{e^{-iv}}{v} \right]$$

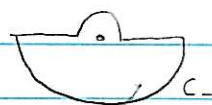
$$= H_- + H_+$$


Complete H_+

Correct since $e^{iv} \rightarrow 0$ as $\text{Im } v \rightarrow +\infty$

Since there are no poles, we have $\oint_{C_+} \frac{e^{iv}}{v} dv = 0$

Complete H_-



correct since $e^{-iv} \rightarrow 0$ as

$$H_- = -\frac{1}{4i} \int_{C_-} \frac{e^{-iv}}{v} dv$$

$$= -\frac{1}{4i} (-2\pi i) = \pi/2$$

Finally $G(\vec{r}) = G(r) = -\frac{1}{4\pi r}$

Then is totally expected from concepts of physics. It says that the electric potential of a unit pt. charge at the origin is $\sim 1/r$.

Ex: Gradient of any $f(r)$ is $\vec{\nabla} f(r) = \hat{x} f'(r)$ *

Electric field $\vec{E} = -\vec{\nabla} G(r) = +\frac{1}{4\pi} \frac{\hat{x}}{r^2}$

Inverse square law force.

We can use our $G(\vec{x}-\vec{y})$ to write sol of Poisson eq,

$\nabla^2 u(\vec{x}) = \rho(\vec{x})$ for any source ρ as

$$u(\vec{x}) = -\frac{1}{4\pi} \int d^3y \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$$

i) integral converges as $\vec{y} \rightarrow \vec{x}$ since $\frac{1}{|\vec{x}-\vec{y}|}$ is an integrable sing. for $D=3$. In sph coords centered at $\vec{x}$,

$$\int d^3y \frac{1}{|\vec{x}-\vec{y}|} \sim \int \frac{r^2 dr}{r} \leftarrow \text{well defined.}$$

ii) it converges as $\vec{y} \rightarrow \infty$ if $\rho(\vec{y})$ vanishes suff. fast. We want a situation when total charge or mass is finite

$$\int d^3y \rho(\vec{y}) = \int r^2 dr d(\cos\theta) d\phi \rho(\vec{y}) < \infty$$

Converges if $|\rho(\vec{y})| \lesssim \frac{1}{|\vec{y}|^{3+\epsilon}}$

Then $\int d^3y \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$ converges even faster.

Emphasize $G = -\frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{y}|}$ is both trans. + rotation invar.

because $\mathbb{R}^3$ is both homogeneous and isotropic.

A simpler way to say this: the response at $\vec{x}$ to a pt. charge at $\vec{y}$ depends only distance $|\vec{x}-\vec{y}|$.

Study an example of Poisson's eq (std. in physics courses)

$$\nabla^2 u(\vec{x}) = \begin{cases} \rho(r) & r < R \\ 0 & r > R \end{cases} \quad \text{bounded sph. sym source.}$$

Equip. phys. question: what is grav. potential due to mass distrib. inside the earth.



Write $\vec{x} = r \hat{x}$ $\hat{x}^2 = \hat{y}^2 = 1$

$\vec{y} = r' \hat{y}$ $|\vec{x} - \vec{y}| = (r^2 + r'^2 - 2rr' \hat{x} \cdot \hat{y})^{1/2}$

The Green's fn sol. is.

$$u(\vec{x}) = -\frac{1}{4\pi} \int_{r' \leq R} d^3y \frac{\rho(r')}{(r^2 + r'^2 - 2rr' \hat{x} \cdot \hat{y})^{1/2}}$$

There are techniques to do this triple integral, but it is more intelligent to solve PDE indirectly using

i) sph. sym of problem

ii) Gauss' Thm $\int_V d^3x \vec{\nabla} \cdot \vec{F} = \int_S d^2x \hat{n} \cdot \vec{F}$

Sph sym $\Rightarrow u(\vec{x}) = u(r)$

(since there is no preferred direction, the potential only depends on distance from origin)

Let $\vec{F} = \vec{\nabla} u$ so PDE is

$$\vec{\nabla} \cdot \vec{F} = \begin{cases} \rho(r) & r < R \\ 0 & r > R \end{cases}$$

For any V bounded by surface S

$$\int_S d^2x \hat{n} \cdot \vec{\nabla} u = \int_V d^3x \vec{\nabla} \cdot \vec{F}$$

$$= \int_V d^3x \rho(r) \Theta(R-r)$$



← step for $\Theta(R-r)$



Most useful choice is to take V to be a ball with center at origin, radius r_0

$$\hookrightarrow \int_0^{r_0} r^2 dr \rho(r) \Theta(R-r) \int_{\Omega} \sin \theta d\theta d\phi$$

$$\int_{\Omega} = 4\pi$$

$$= M(r_0) \leftarrow \text{mass contained in sph. of radius } r_0$$

if $r_0 > R$, then $M(r_0) = M = \text{total mass}$

Now we need to evaluate surface integ.

From Ex. *

a) $\hat{n} \cdot \vec{\nabla} u = \hat{n} \cdot \hat{x} u'(r) = u'(r)$

b) use sph coords $d^2x = r_0^2 \sin \theta d\theta d\phi$

c) $\int_S d^2x \hat{n} \cdot \vec{\nabla} u = r_0^2 u'(r_0) = M(r_0)$

Then $u'(r_0) = \frac{M(r_0)}{4\pi r_0^2}$, since r_0 is arbit., we will let $r_0 \rightarrow r$ and write result $u'(r) = \frac{M(r)}{4\pi r^2}$

This means that grav force $\vec{F} = -\nabla u = -\hat{x} u'(r)$ is

$$\vec{F} = -\frac{1}{4\pi} \frac{M(r)}{r^2} \hat{x}$$

c) if $r > R$ $M(r) = M_{\text{earth}}$. Outside the earth we

find exact $1/r^2$ attractive inverse sq. law force of Newtonian gravity.

d) if $r < R$, the attraction decreases as we move inward.

The force one feels at any r is attractive $1/r^2$ force due to the amt. of mass between obs pt r and origin. (The net force due to mass at $r_0 < r < R$ cancels).

Another receipt on Green fn for ∇^2 in $\mathbb{R}^d$. We will derive result by a very simple method which works for all d . Illustrates power of symmetry + dimen. analysis.

Prob: find sol of $\nabla_x^2 G^{(d)}(\vec{x} - \vec{y}) = \delta^{(d)}(\vec{x} - \vec{y})$ in $\mathbb{R}^d$

a) Use trans sym to set $\vec{y} = 0$. $\rightarrow \nabla^2 G^{(d)}(\vec{x}) = \delta^{(d)}(\vec{x})$

b) there are no dimensionful parameters in this eqn. (in fact no params. at all). We must construct sol only in terms of coords $x_1, x_2, \dots, x_d$.

c) rotational sym $\Rightarrow G$ can only depend on distance from origin $r = \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$

d) some dimensional analysis $[\nabla^2] = 1/l^2$

$$[\delta^{(d)}(\vec{x})] = 1/l^d \quad \text{since} \quad \int d^d k \delta^{(d)}(\vec{x}) = 1$$

From PDE $\nabla^2 G = \delta$ we find $[G^{(d)}] = 1/l^{d-2}$

e) So the Grn. fn is a fn of r of dim $1/l^{d-2}$

For $d > 2$, the only possibility is $G^{(d)}(\vec{x}) = c_d / r^{d-2}$

So sym. and dim. analy. has determined Gr. fn up to a const. factor c_d .

What about $d=2$, dim. analysis allows $G^{(2)} = c$, but then $\nabla^2 G^{(2)} \equiv 0$ not $\delta^{(2)}(\vec{x})$.

We need something else. Look at $|\vec{\nabla} G^{(2)}(\vec{x})|$, rather than $G^{(2)}$.

Apply sym + dimen. args. to $\vec{\nabla} G^{(2)}$:

$$\text{PDE} \quad \nabla \cdot (\nabla G^{(2)}) = \delta^{(2)}$$

$$\frac{1}{\ell} \quad \frac{1}{\ell} = 1/\ell^2 \Rightarrow [\vec{\nabla} G^{(2)}] = \frac{1}{\ell}$$

Rotational sym $\Rightarrow G^{(2)} = f(r)$,

$$\text{then } \vec{\nabla} G^{(2)} = \frac{df}{dr} \vec{\nabla} r = f'(r) \hat{x} \quad \text{as in Ex 8}$$

Since $[f'] = 1/\ell$, we have scal. $f'(r) = c_2/r$

Then $f(r) = c_2 \ln(r/r_0)$. Think of r_0 as

an "integ. const" necessary to make argument of $\ln()$ dimensionless. We will discuss this soon.

How do we find the constants c_d ? apply Gauss law to a ball of radius R about origin.

$$\int_{V_d} d^d x \vec{\nabla} \cdot \vec{F} = \int_{S_{d-1}} d^{d-1} x \hat{n} \cdot \vec{F} \quad \text{take } F = \nabla G$$

$$\text{lhs} = \int_{V_d} d^d x \nabla^2 G = \int d^d x \delta^d(x) = 1$$

$$\begin{aligned} \text{rhs} \quad \hat{n} \cdot \nabla G &= G'(r) = c_d \frac{\partial}{\partial r} \begin{cases} 1/r^{d-2} & d > 2 \\ \ln r/r_0 & d = 2 \end{cases} \\ &= c_d \begin{cases} -(d-2)/r^{d-1} & d > 2 \\ 1/r & d = 2 \end{cases} \end{aligned}$$

$$\text{So rhs} = c_d \frac{1}{R^{d-1}} \int_{S_{d-1}} d^{d-1} x \cdot \begin{cases} -(d-2) \\ 1 \end{cases}$$

$$\text{Def: } \Omega_d = \frac{1}{R^{d-1}} \int_{S_{d-1}} d^{d-1} x = \text{surf area of unit sph in } d \text{ dim.}$$

$$\text{For } d=2 \quad \text{we know } \Omega_2 = 2\pi$$

$$d=3 \quad \text{" " " " } \Omega_3 = 4\pi$$

For general d : $\Omega_d = \frac{2\pi^{d/2}}{r^{(d/2)}} \leftarrow \text{can be obtained from integral of } e^{-\vec{x} \cdot \vec{x}} \text{ in } d\text{-dim.}$

Conclusion from Gauss law

$$\text{lhs} = \text{rhs:} \quad 1 = c_d \Omega_d \begin{cases} -(d-2) & d > 2 \\ 1 & d = 2 \end{cases}$$

Our final result for Green's fn.

$$d > 2 \quad G^{(d)}(\vec{x}) = \frac{-1}{(d-2)\Omega_d} \frac{1}{r^{d-2}}$$

$$d = 2 \quad G^{(2)}(\vec{x}) = \frac{1}{2\pi} \ln(r/r_0)$$

Using $G^{(d)}$ we can write the sol of $\nabla^2 u = \rho(\vec{x})$ as

$$u(\vec{x}) = \int d^d y \, G^{(d)}(\vec{x} - \vec{y}) \rho(\vec{y})$$

In $d=2$

$$u(\vec{x}) = \frac{1}{2\pi} \int d^2 y \, \rho(\vec{y}) \ln\left(\frac{|\vec{x} - \vec{y}|}{r_0}\right)$$

What's effect of r_0 which is quite artificial + arbitrary.

The r_0 term just gives $-\frac{\ln r_0}{2\pi} \int d^2 y \, \rho(\vec{y})$. It adds a constant to potential $u(\vec{x})$. In physical situation, an additive const in potential has no physical effect. Rather it is the force $\vec{\nabla} u$ which is physical and $\vec{\nabla} u$ does not depend on r_0 .

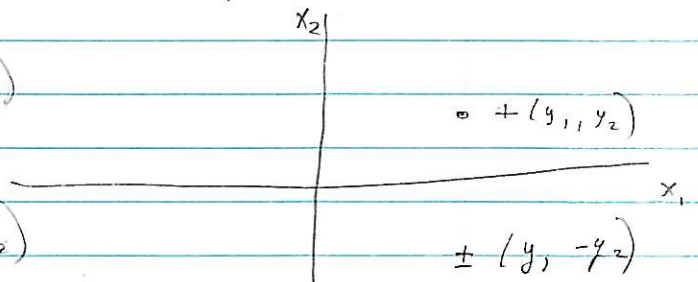
A practical + interesting applic of $G^{(d)}$ is to Image charge methods for $1/2$ space probs.

I. Find $G_{D,N}(x_1, x_2; y_1, y_2)$
on upper $1/2$ plane $x_2 > 0$

$$\nabla^2 G_{D,N}(\vec{x}) = \delta(x_1 - y_1) \delta(x_2 - y_2)$$

$$G_D(x_1, 0; y_1, y_2) = 0$$

$$\frac{\partial}{\partial x_2} G_N(x_1, 0, y_1, y_2) = 0$$



Reason as follows: a) full space $G^2 = \frac{1}{4\pi} \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{r_0^2}$

satisfies PDE with correct pt. source at (y_1, y_2) but not BC

b) add an image charge at $(y_1, -y_2)$ to find complete sol.

$$G_D(x_1, x_2; y_1, y_2) = \frac{1}{4\pi} \left\{ \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{r_0^2} - \ln \frac{(x_1 - y_1)^2 + (x_2 + y_2)^2}{r_0^2} \right\}$$

$$= \frac{1}{4\pi} \ln \frac{(x_1 - y_1)^2 + (x_2 - y_2)^2}{(x_1 - y_1)^2 + (x_2 + y_2)^2} \quad r_0 \text{ drops out.}$$

as $x_2 \rightarrow 0$, $G_D \rightarrow \frac{1}{4\pi} \ln \frac{(x_1 - y_1)^2 + y_2^2}{(x_1 - y_1)^2 + y_2^2} = 0$ so bc is satisfied.

Note that $G_D(\vec{x}, \vec{y}) = G_D(\vec{y}, \vec{x})$ symmetric.

Ex: Work out G_N in same degree of detail:

Prob 9.5.16 of text: Find G for 60° wedge with D bc's

Find arrangement of images by following reasoning.

i) Put usual image with -ve sign at $(y_1, -y_2)$. We now satisfy BC along line $\theta = 0$, but not at $\theta = 60^\circ = \pi/3$.

ii) reflect the prev two sngs about line $\theta = \pi/3$ with -ve sign. We get 4 sources. Satisfies BC at $\theta = \pi/3$, but we lost it at $\theta = 0$.

iii) fix this up by another reflection about $\theta = 0$.

We then have a total of 6 sources, and both bc's are

satisfied! The drawing gives a quite complete answer to the prob. We will not work down the sol in detail: $G = G^2 - G^{(2)} + G^{(2)} - G^2 + G^2 - G^2$

