

Th Oct 29 (after midterm):

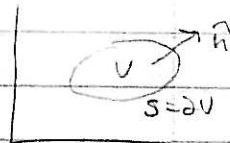
Green's formulas for general $d=2,3,4,\dots$ dimensions.

an important theorem, tool in study of $\nabla^2 u(x) = 0$ $u(x)|_{\partial R} = \phi(x)$ general

and $\nabla^2 \phi_\lambda(x) = -\lambda \phi_\lambda(x)$ $\phi_\lambda(x)|_{\partial R} = 0$

Ref Sec 7.5 of H

To be concrete, consider 3-dim region V ,
bounded by surface $S = \partial V$



1. Gauss' Theorem or divergence theorem for this situation.

For any vector field $\vec{F}(\vec{x})$ with contin. 1st partial derivs in V

$$\int_V d^3x \vec{\nabla} \cdot \vec{F} = \int_S d^2x \hat{n} \cdot \vec{F}$$

In words: Vol. integral of divergence = surface integral of flux.

Comments: i. This is important, but it is really a multi-dim extension of fund. thm. of calculus.

See ref for $\square$ on p 38 of notes.

(ii) d^3x = vol element in V . In Cartesian coords. x_1, x_2, x_3

$$d^3x = dx_1 dx_2 dx_3 \quad \text{and} \quad \vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \frac{\partial}{\partial x_3} F_3$$

For many probs, another set of coords is more convenient. We must then change variables carefully.

Same is true for description of surface S .

2. Green's 1st identity: let $u(\vec{x}), v(\vec{x})$ be two scalar fns. with contin. 2nd partial derivs in V . Take $\vec{F} = v \vec{\nabla} u$.

Gauss $\Rightarrow$

$$\begin{aligned} \int_V d^3x \vec{\nabla} \cdot (v \vec{\nabla} u) &= \int_V d^3x [\nabla v \cdot \nabla u + v \nabla^2 u] \\ &= \int_S d^2x v \hat{n} \cdot \vec{\nabla} u \end{aligned}$$

Similar structure to the mult+ integ arg we used earlier for 1-dim probs.

3. Green's 2nd identity (Green's formula) Rewrite Green, with $v \leftrightarrow u$. Subtract the two expressions. Note that $\nabla v \cdot \nabla u$ is sym. in $u \leftrightarrow v$ and cancels.

Important result: $\int_V d^3x [v \nabla^2 u - u \nabla^2 v] = \int_S d^2x [v \hat{n} \cdot \vec{\nabla} u - u \hat{n} \cdot \vec{\nabla} v]$

i) Green₁ and Green₂ are identities which hold for any pair of fns $u(\vec{r}), v(\vec{r})$ (with contin 2nd partial derivs)

ii) The identrs. have many applics. For each one uses an approp. choice of u, v .

Apply to eigenval prob: $\nabla^2 \phi = -\lambda \phi$ in V with homog.

D or N BC's on ∂V .

a) take $u=v=\phi(\vec{r})$ in Green₁.

$$\int_V d^3x [\nabla \phi \cdot \nabla \phi + \underbrace{\phi \nabla^2 \phi}_{-\lambda \phi^2 \text{ by PDE}}] = \int_{\partial V} d^2x \underbrace{\phi \hat{n} \cdot \vec{\nabla} \phi}_{=0 \text{ by BC's}}$$

$$\text{Then } \lambda \int_V d^3x \phi^2 = \int_V d^3x \nabla \phi \cdot \nabla \phi = \int_V d^3x (\nabla \phi)^2 \geq 0$$

Hence $\lambda \geq 0$ and eigenvalues are non negative.

Ex: Prove $\lambda > 0$ for D BC's, and that $\lambda = 0$ always occurs for N BC's

b) take $u=\phi_1, \nabla^2 \phi_1 = -\lambda_1 \phi_1$ use Green₂
 $v=\phi_2, \nabla^2 \phi_2 = -\lambda_2 \phi_2$

$$\int_V d^3x [\phi_2 \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2] = \int_{\partial V} d^2x (\phi_2 \hat{n} \cdot \vec{\nabla} \phi_1 - \phi_1 \hat{n} \cdot \vec{\nabla} \phi_2) = 0 \text{ BC's}$$

$$\Rightarrow (\lambda_2 - \lambda_1) \int_V d^3x \phi_1 \phi_2 = 0$$

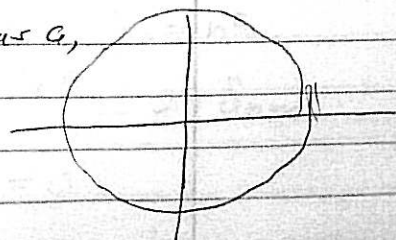
So eigenfns of distinct λ_n are orthog

(It's all easy with the right tools for the task)

Illustrative Exmplo of Lap. / D prob

Sec 2.5.2

Find sol of $\nabla^2 u = 0$ on a disc of radius a ,
 with BC. $u(a, \theta) = f(\theta)$



Because of circular sym of prob, we use
 polar coords $x = r \cos \theta$
 $y = r \sin \theta$

See prob 1.5.3 of H (p. 28) where you are guided to a demonstr. that $\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0$ in Cartesian coords, is equiv to $\nabla^2 u(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

In addition to physical b.c at $r=a$, we need cond at the artificial bds. introduced in polar coords.

- (i) We have $-\pi \leq \theta \leq \pi$, but $\pm \pi$ describe the same physical angle. So we must impose period. BC's $u(r, \pi) = u(r, -\pi)$
 $\frac{\partial}{\partial \theta} u(r, \pi) = \frac{\partial}{\partial \theta} u(r, -\pi)$
- (ii) the origin $r=0$ is also a mathem. bdy since $0 \leq r \leq a$. But there is an ordinary physical pt, so we require that $\lim_{r \rightarrow 0} u(r, \theta)$ exists.

Solve this prob. by sep. of vars.

- (a) Look for prod. sols: $u(r, \theta) = \phi(\theta) G(r)$. Subst in PDE to get $\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) \phi(\theta) + \frac{1}{r^2} G(r) \phi''(\theta) = 0$

Mult by $\frac{r^2}{\phi(\theta) G(r)}$ to get.

$$-\frac{1}{G(r)} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = \frac{1}{\phi(\theta)} \phi''(\theta) = -\lambda$$

So we derive 2 ODE's: $\phi''(\theta) = -\lambda \phi(\theta) \quad -\pi < \theta < \pi$
 and $r \frac{d}{dr} \left(r \frac{dG}{dr} \right) = -\lambda G(r)$ PBC's

A pair of eigen. probs. linked by common eigen. λ .

We know the soln. of θ prob. from our work on cr. form of F. series. $\exists$ a complete set of cr. eigenfns

$$\phi_n(\theta) = e^{in\theta} \quad -\infty < n < \infty \quad \lambda_n = n^2$$

$$\text{orthog: } (\phi_n, \phi_m) = \int_{-\pi}^{\pi} d\theta e^{-in\theta} e^{im\theta} = 2\pi \delta_{nm}$$

The r eqn is $r^2 G'' + r G' - n^2 G = 0$.

This is just the Cauchy-Euler eq from 18.3.

General sol: $G(r) = c_1 r^{|n|} + c_2 r^{-|n|} \quad n \neq 0$
 $G(r) = c_1 + c_2 \ln r \quad n = 0$

The regularity cond. at $r=0$ forces $c_2=0$. So we are left with the sep. sols:

$$u_n(r, \theta) = r^{|n|} e^{in\theta} \quad -\infty < n < \infty$$

Since the PDE is linear any superposition of these is also a sol at $\nabla u = 0$, so the general is

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} \tilde{c}_n r^{|n|} e^{in\theta}$$

The only remaining issue is to satisfy b.c. $u(a, \theta) = f(\theta)$. We assume that $f(\theta)$ is smooth enough to have a converg.

F. series $f(\theta) = \sum_n c_n e^{in\theta}$ with coeffs.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta' e^{-in\theta'} f(\theta')$$

If we choose $\tilde{c}_n = \frac{c_n}{a^{|n|}}$ we have the sol of full prob

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{r}{a}\right)^{|n|} e^{in\theta}$$

satisfies both
PDE + BC.

Convergence of series? $|u| \leq |c_0| + \sum_{n=1}^{\infty} (|c_n| + |c_{-n}|) \left(\frac{r}{a}\right)^n$

Since $|c_n| \rightarrow 0$ at least as fast as $1/n$, the power series in r has radius of conv. $R \geq a$ (typically $R=a$)

The sol. is a smooth fn in interior of disc.

Next step is to go Green: substit. formula for c_n into the Fourier sum and exch. $\sum_n \leftrightarrow \int d\theta'$

$$\begin{aligned} u(r, \theta) &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} d\theta' \frac{1}{2\pi} \left(\frac{r}{a}\right)^{|n|} e^{in(\theta-\theta')} f(\theta') = \int_0^{2\pi} \text{same} \\ &= \int_0^{2\pi} d\theta' g(r, \theta-\theta') f(\theta') \end{aligned}$$

with greenish fn $g(r, \theta-\theta') = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{r}{a}\right)^{|n|} e^{in(\theta-\theta')}$

Comments: 1) "universal" form which shows how $u(r, \theta)$ in interior is determined by any chosen bdy data.

2) g is called the Green's fn. b/c it is related to, but not same as the true Green's fn G for the Laplace ∇^2 . We will study G later. The proper name of g is Poisson Kernel.

3) Note that $g(r, \theta - \theta')$ is a fn of the diff. $\theta - \theta'$ because of the translation sym along the circum. of the disc.

Final step: sum the series to put g in "closed form".

Let $z = \frac{r}{a} e^{i(\theta - \theta')}$ note $|z| < 1$ in interior

$$g = \frac{1}{2\pi} \left\{ 1 + \sum_{n=1}^{\infty} (z^n + \bar{z}^n) \right\}$$

$$= \frac{1}{2\pi} \left\{ 1 + \left(\frac{1}{1-z} - 1 \right) + \left(\frac{1}{1-\bar{z}} - 1 \right) \right\}$$

Ex: $= \frac{1}{2\pi} \frac{1 - z\bar{z}}{1 - z - \bar{z} + z\bar{z}}$ straightforward algebra.

Go back to real variables $z + \bar{z} = \frac{2r}{a} \cos n(\theta - \theta')$

$$z\bar{z} = r^2/a^2$$

Ex: Final Form is

$$g(r, \theta - \theta') = \frac{1}{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta - \theta')}$$

So the sol of the Laplace/D prob on $\bigcirc$ with $u(r, \theta) \rightarrow f(\theta)$ as $r \rightarrow a$ can be expressed as Poisson integral formula:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta' \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta - \theta')} f(\theta')$$

Suppose you were presented with the formula without the long derivation. You might ask two practical questions

1. Why does $u(r, \theta)$ satisfy the PDE $\nabla^2 u = 0$?
2. Why does it satisfy BC.

Answer to 1: $\nabla^2 u = 0$

$$\nabla^2 u(r, \theta) = \int_0^{2\pi} d\theta' \left[\nabla_{r, \theta}^2 g(r, \theta - \theta') \right] f(\theta') \stackrel{?}{=} 0$$

sol satisfies $\nabla^2 u = 0$ if $\nabla^2 g = 0$, that is if g

is a harmonic fn in the obs. variables r, θ .

This can probably be checked using the form of $\nabla^2_{r, \theta}$ on p 68, but it is simpler to go back to the formula

$$2\pi g = -1 + \frac{1}{1-z} + \frac{1}{1-z^*} = -1 + 2 \operatorname{Re} \frac{1}{1-z}$$

$$\text{Let } \beta = \frac{1}{\alpha} e^{-i\theta'} \quad re^{i\theta} = x + iy$$

$$\text{Then } \frac{1}{1-z} = \frac{1}{1-\beta(x+iy)}, \text{ so } g \text{ contains}$$

the real part of analytic fn. (a fn of $x+iy$)

A basic result of 18.04, real part of any $F(x+iy)$ is automatically a harmonic fn.

$$\text{i.e. } \nabla^2 \operatorname{Re} F(x+iy) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \operatorname{Re} F = 0$$

Ex: Prove this using chain rule & outline below

$$\text{Think of } F(x+iy) = F(z) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} \quad \text{power series}$$

$$\frac{d^n}{dz^n} F = c_n$$

$$\frac{\partial}{\partial x} F(z) = F'(z) \quad \frac{\partial}{\partial y} F(z) = i F'(z)$$

$$\text{Then } \frac{\partial^2}{\partial x^2} F = F'' \quad \frac{\partial^2}{\partial y^2} F = -F''$$

$$\text{Hence } \nabla^2 F = 0.$$

$$\text{Same is true for } \bar{F} = \sum_{n=0}^{\infty} \bar{c}_n \frac{\bar{z}^n}{n!} \quad \bar{z} = x - iy$$

$$\text{So } \nabla^2 \operatorname{Re} F = \frac{1}{2} \nabla^2 (F + \bar{F}) = 0$$

What about the BC? Consider

$$\lim_{r \rightarrow a} u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\theta' f(\theta') \lim_{r \rightarrow a} \frac{a^2 - r^2}{a^2 + r^2 - 2ar \cos(\theta - \theta')}$$

Look at properties of

$$g = \frac{1}{2\pi} \frac{a^2 - r^2}{(a-r)^2 + 2ar(1 - \cos(\theta - \theta'))} \quad \text{in lim } r \rightarrow a$$

c) clearly $g \geq 0$ in $r \leq a$

(c) if $\theta \neq \theta'$ $g(r, \theta - \theta') \xrightarrow{r \rightarrow a} 0$

if $\theta = \theta'$ $g(r, \theta) = \frac{1}{2\pi} \frac{a+r}{a-r} \xrightarrow{r \rightarrow a} \infty$

We now suspect that $\lim_{r \rightarrow a} g(r, \theta - \theta') = c \delta(\theta - \theta')$

To test this and to find the const. c , look at

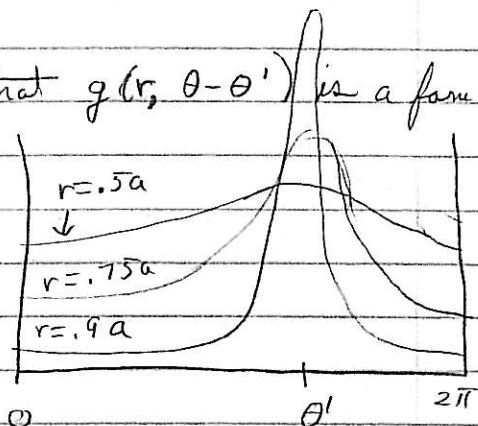
$$\int_0^{2\pi} d\theta \frac{1}{a^2 + r^2 - 2ar \cos \theta} = \frac{2\pi}{a^2 - r^2} \quad \text{if } r < a$$

$\left\{ \begin{array}{l} \text{Gradshteyn +} \\ \text{Ryzhik} \\ 3.612 \# 2 \end{array} \right.$

So $\frac{a^2 - r^2}{2\pi} \int \text{same} = \int_0^{2\pi} d\theta g(r, \theta) = 1 \quad \text{if } r < a$

So $c = 1$ above.

Thus is enough info to see that $g(r, \theta - \theta')$ is a family of smooth fns of θ , which become more and more sharply peaked about $\theta = \theta'$, as $r \rightarrow a$



Next subject: uniqueness of solns of Laplace eqn in finite regions. There is a simple approach to I/D problem using the maximum-minimum principle. See sec 2.5.4 of H. In class we will prove uniqueness as an applic of Green's formulas.