

Note that zeros of $\phi_{nm}(x)$ are along lines, not isolated pts.

$$\text{Zeros of } \phi_{32} = \sin 3x \sin 2y$$

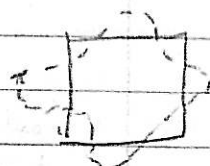
$$\lambda = 3^2 + 2^2 = 13$$



The cond for a zero of a fn in $d=2$ is $f(x,y)=0$.

Generically, the 'zero set' is a 1-dim curve in the plane.

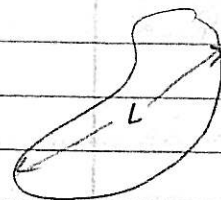
No surprise that same is true for eigenfns. The same feature should remain true for Dirichlet eigenfns on regions which are smooth deformations of square.



An interesting semi-quant, arg. to estimate lowest non-zero eigenvalue λ_1 for D prob on general R . We are looking for sol of $\nabla^2 \phi_1(\vec{x}) = -\lambda_1 \phi_1(\vec{x})$

$$\phi_1(x) \Big|_{\partial R} = 0$$

Define spatial scale L = typic linear size of region



For heat flow prob on some R , the lowest sep. sol is

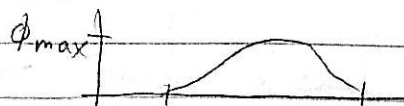
$u_1(x,t) = e^{-\lambda_1 t} \phi_1(\vec{x})$. So the time scale of the slowest mode is $T_1 = 1/\lambda_1$. But we know for diffusion, that $T_1 = \frac{1}{c} L^2$, so we expect $\lambda_1 \approx \frac{c}{L^2}$

We want to estimate the value of c .

We can assume that $\phi_1(\vec{x})$ is normalized $\iint_R \phi_1^2 = 1$

$\phi_1(x)$ vanishes on ∂R . Along any line across R ,

$\phi_1(x)$ looks like



Estimate $\phi_{\max}$ from normalization integral:

$$1 = A \alpha \phi_{\max}^2$$

where α is some fraction, e.g. $\alpha \approx 1/2$, but not precise

$$\text{So we obtain } \phi_{\max} = \frac{1}{\sqrt{\alpha A}}$$

From eigen. problem: $\lambda_1 \phi_1 = -\nabla^2 \phi_1$

$$(f, g) \equiv \int_{\mathcal{R}} d^2x \phi(x) g(x)$$

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Multip by $\phi_1(x)$ and integrate

$$\begin{aligned} \lambda_1 (\phi_1, \phi_1) &= - \int_{\mathcal{R}} d^2x \phi_1(x) \nabla^2 \phi_1 \\ &\equiv + \int_{\mathcal{R}} d^2x \vec{\nabla} \phi_1 \cdot \vec{\nabla} \phi_1 \\ &\quad - \int_{\partial \mathcal{R}} d\mathbf{x} \phi_1 \cdot \mathbf{n} \cdot \vec{\nabla} \phi_1 \end{aligned}$$

by partial int
Green's 1st
identity, TBA
H, p 295

So we have (since $(\phi_1, \phi_1) = 1$ normalized)

$$\lambda_1 = \int_{\mathcal{R}} d^2x (\vec{\nabla} \phi_1)^2$$

Eigenvalue λ_n is mean value
of $(\vec{\nabla} \phi_n)^2$, a general
and exact result for
normalized eigenfn.



the plot to estimate $|\vec{\nabla} \phi_1|$. ϕ_1 goes from 0 to $\phi_{\max}$ over an interval of length $L/2$.

$$\text{So } |\vec{\nabla} \phi_1| \approx \frac{|\phi_{\max}|}{L/2} = \frac{2\phi_{\max}}{L} \quad \text{Then our estimate}$$

$$\text{of the eigenvalue } \lambda_1 \approx A \cdot 2 \cdot \left(\frac{2\phi_{\max}}{L} \right)^2$$

since $\vec{\nabla} \cdot \phi \cdot \vec{\nabla} \phi = (\partial_x \phi)^2 + (\partial_y \phi)^2$

$$\lambda_1 \approx 8 A \left(\frac{\phi_{\max}}{L} \right)^2 \approx 8 \frac{1}{\alpha L^2}$$

← agrees with our
expec. based on scalar
in diffusion.

Ex: For a d-dim region, the same arg. gives $\lambda_1 \approx \frac{4d}{\alpha L^2}$

How good is the estimate for π  ?

We know $\phi_1 \rightarrow \phi_{1,1} \equiv \frac{1}{\pi} \sin x \sin y$ and $\lambda_{1,1} = 2$

Rewrite our estimate λ_1 as $\alpha \approx \frac{8}{\lambda_{1,1} L^2} = \frac{8}{2\pi^2} \approx .406$

A very reasonable value for the fraction α in fact part of
our estimate.

D

Go back top p. 59 and complete sol of heat flow on $\mathbb{R}$.

$$\partial_t u = \nabla^2 u \quad \vec{x} \in \mathbb{R} \quad u=0|_{\partial\mathbb{R}} \quad u(x,0) = f(\vec{x})$$

The separable sols are $u_n(\vec{x}, t) = e^{-\lambda_n t} \phi_n(\vec{x})$

Step ③: The mathematical theory guarantees that there is an eigenfnl expansion:

$$u(\vec{x}, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \phi_n(\vec{x})$$

with $\{a_n\}$ determined by initial data: $a_n = \frac{1}{N_n^2} (\phi_n, f)$

$$= \frac{1}{N_n^2} \iint d^2 y \phi_n(\vec{x}) f(\vec{y})$$

Convergence: superfast for $t > 0$

conv. at $t=0$ depends how "smooth" $f(\vec{x})$ is.

For $t > 0$ $u(\vec{x}, t)$ will be super smooth.

Step ④: subst and exchange $\int \leftrightarrow \iint$

$$u(\vec{x}, t) = \iint d^2 y \sum_{n=1}^{\infty} \frac{e^{-\lambda_n t} \phi_n(\vec{x}) \phi_n(\vec{y})}{N_n^2} f(\vec{y})$$

Identify

$G(\vec{x}, \vec{y}, t) \leftarrow$ Green's fn for heat flow in $d=2$ in spectral form!

Ex: Generalize arg. on p 46 of notes to formulate

IVP for $G(\vec{x}, \vec{y}, t)$: $\frac{\partial}{\partial t} G(\vec{x}, \vec{y}, t) = \nabla_{\vec{x}}^2 G(\vec{x}, \vec{y}, t)$

$$G(\vec{x}, \vec{y}, t) \Big|_{\vec{x} \in \mathbb{R}} = 0$$

$$\lim_{t \rightarrow 0^+} G(\vec{x}, \vec{y}, t) = \delta^{(2)}(\vec{x} - \vec{y})$$

Ref: H. Sec 2.5

Message: heat flow for $d=2$ has same "structure" as for $d=1$. Important for students to understand this "structure!"

Now consider heat flow on $\mathbb{R}$ with non-trivial

bdy temp $u(\vec{x}, t) \Big|_{\vec{x} \in \partial\mathbb{R}} = T(\vec{x})$

$\mathbb{R}$ prescribed bdy temp.

indep of t

Physical?: What is long time limit of $u(\vec{x}, t)$. We expect steady state or equil dist. $u(\vec{x}, t) \xrightarrow{t \rightarrow \infty} u_s(\vec{x})$

In this lim, the heat eq $\partial_t u = \nabla^2 u$ becomes

$$\boxed{0 = \nabla^2 u_s(\vec{x}) \quad u_s(\vec{x})|_{\vec{x} \in \partial R} = T(\vec{x})}$$

Summary: the long time limit of D heat flow on R is a steady state dist $u_s(\vec{x})$ which satisfies Laplace eqn on R with non-trivial bdy data.

(This is not the same prob as $\nabla^2 \phi_n = -\lambda_n \phi_n$ with $\phi_n|_{\partial R} = 0$)

Laplace's eqn occurs in more fundamental situations; such as dist. of ^{charged} conductors in 3-dim space (90) (91)

Electric field $\vec{E} = -\vec{\nabla} V(\vec{x})$

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 V(\vec{x}) = 0 \quad \text{with certain BC's} \quad (92)$$

$V(\vec{x})$ is electrostatic potential

which we will not discuss

(1) The full sol of $\nabla^2 u = 0$ heat flow problem is:

$$u(\vec{x}, t) = u_s(\vec{x}) + \iint_R d^2y G_D(\vec{x}, y, t) f(y)$$

$$(2) \text{ Laplace } \nabla^2 u = 0 \quad R \\ = u_s(\vec{x}) + u_{G_D}(\vec{x}, t)$$

in which:

(i) $u_s(\vec{x})$ 'captures' long time limit of sol and is determined by bdy data $T(\vec{x})$, $\vec{x} \in \partial R$

(ii) $u_{G_D} \rightarrow 0$ as $t \rightarrow \infty$ determined by initial data, $f(\vec{x})$, $\vec{x} \in R$

Neumann prob: $\partial_{\vec{x}} u = \nabla^2 u \quad \vec{x} \in R$

$$\text{BC: } -\hat{n} \cdot \vec{\nabla} u|_{\partial R} = \Phi(\vec{x}) \quad \leftarrow \text{bdy heat flux..}$$

$$\text{IC: } u(\vec{x}, 0) = f(\vec{x})$$

Physically we expect steady state $u(\vec{x}, t) \xrightarrow{t \rightarrow \infty} u_s(\vec{x})$ only if $\oint$ no net heat flux into R

$$\oint_{\partial R} dx \Phi(\vec{x}) = 0$$

From lim of PDE $\lim_{t \rightarrow \infty}$

$$0 = \nabla^2 u_s \quad \text{with } -\hat{n} \cdot \vec{\nabla} u|_{\partial R} = \Phi(\vec{x})$$

This is the Laplace/Neumann problem, a curious fact.

There is no sol unless $\oint_{\partial R} \Phi(\vec{x}) = 0$

pf using Gauss law

$$0 = \iint_R d^2x \vec{\nabla} \cdot \vec{\nabla} u_s = \int_{\partial R} dx \hat{n} \cdot \vec{\nabla} u_s = - \int_{\partial R} dx \Phi(\vec{x})$$

Very interesting that simple mathematical fact about Laplace eqn. agrees with physical intuition from heat flow.

Ex: The thermal energy $E = \iint_R d^2x u(\vec{x}, t)$ satisfy

$$\frac{dE}{dt} = \iint_R d^2x \nabla^2 u = - \int_{\partial R} dx \Phi(\vec{x}) = 0 \quad \leftarrow \text{assume}$$

Ex: In this situation the full sol of N heat problem is

$$u(\vec{x}, t) = u_s(\vec{x}) + u_{G_N}(\vec{x}, t)$$

where $u_s(\vec{x})$ is long time limit determined by flux $\Phi(\vec{x})$ which must satisfy $\int_{\partial R} dx \Phi(\vec{x}) = 0$

and

$$u_{G_N}(\vec{x}, t) = \iint_R d^2y G_N(\vec{x}, \vec{y}, t) f(\vec{y})$$

$$G_N(\vec{x}, \vec{y}, t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_n t}}{N_n^2} \phi_n(\vec{x}) \phi_n(\vec{y})$$

The eigenfns $\phi_n(\vec{x})$ satisfy $\nabla^2 \phi_n(\vec{x}) = -\lambda_n \phi_n$
 $\hat{n} \cdot \nabla \phi_n \Big|_{\partial R} = 0$

The $\lambda_n \geq 0$ and there is a zero mode $\lambda_0 = 0$ with $\hat{\phi}_0(\vec{x}) = \frac{1}{\sqrt{A}}$

Long time limit $\lim_{t \rightarrow \infty} u_{G_N}(\vec{x}, t) = \frac{1}{A} \iint_R d^2y f(\vec{y})$

mean value of initial temp.

We have reasoned our way to the sol of Neumann heat prob. by recognizing that it has similar "structure" to the D prob with the simple new feature of a zero mode.