

$$\partial_t \bar{u} - v_0 \partial_y \bar{u} = \partial_y^2 \bar{u} - 2\alpha \partial_y \bar{u} \quad \text{cancel if } v_0 = 2\alpha$$

So the new PDE just describes ordinary heat flow in the moving spatial coord y .

This description is correct for heat flow on ∞ line, but not if there are fixed bdy's in original coord. x .

Set up bdy value prob for same PDE on $[0, \pi]$

$$\partial_t u = \partial_x^2 u - 2\alpha \partial_x u$$

$$\text{BC Neumann } \partial_x u(0, t) = 0 \quad \partial_x u(\pi, t) = 0$$

Ex: Is total thermal energy (or total amt of diffusing substance) conserved $\frac{d}{dt} \int_0^\pi dx u(x, t) \stackrel{?}{=} 0$ ← seemingly not

Apply good old method of sep. of vars: $u(x, t) = T(t) \phi(x)$

$$\frac{\dot{T}(t)}{T(t)} = \frac{\phi''(x) - 2\alpha \phi'(x)}{\phi(x)} = -\lambda$$

We find spatial eigen. prob $\phi'' - 2\alpha \phi' = -\lambda \phi$. This is sort of S-L type but it can be made so using integ. factor $e^{-2\alpha x}$

$$e^{-2\alpha x} \phi'' - 2\alpha e^{-2\alpha x} \phi' = -\lambda e^{-2\alpha x} \phi$$

$$\frac{d}{dx} \left(e^{-2\alpha x} \frac{d\phi}{dx} \right) = -\lambda e^{-2\alpha x} \phi$$

$$\text{orig. BC's } \phi'(0) = \phi'(\pi) = 0$$

(We could have done this before sep of vars; The PDE

$$e^{-2\alpha x} \partial_t u = \frac{\partial^2}{\partial x^2} \left(e^{-2\alpha x} \frac{\partial u}{\partial x} \right) \quad \text{This looks like heat}$$

flow in a non-uniform material with exponential density + conductivity)

Apply our early work on S-L probz to * [

i) Eigenfn's ϕ_n satisfy weighted orthog cond.

$$(\phi_n, \phi_m) \equiv \int_0^\pi dx e^{-2\alpha x} \phi_n(x) \phi_m(x) = N_n^2 \delta_{nm}$$

$$\text{ii) } \lambda_n \geq 0$$

iii) $\lambda_0 = 0$ with $\phi_0(x) = 1$ occurs in the Neumann problem.

Find $\phi_n(x)$ by 18.03 guess $\phi = e^{rx}$ in either $\checkmark$ or $\times$ form of ODE

$$\text{Charact. Polyn } q^2 - 2\alpha q + \lambda = 0$$

$$(q - \alpha)^2 + \lambda - \alpha^2 = 0 \quad \text{roots } q = \alpha \pm \sqrt{\lambda - \alpha^2}$$

The general soln
 $\phi = e^{\alpha x} (c_1 e^{i\sqrt{\lambda-\alpha^2}x} + c_2 e^{-i\sqrt{\lambda-\alpha^2}x})$

Ex: one soln $\phi_0(x)=1$ for $\lambda_0=0$, this is the expected Neumann zero mode

Impose BC's to find remaining $\lambda_n, \phi_n(x)$:

$$\phi' = e^{\alpha x} [c_1 (\alpha + i\sqrt{\lambda-\alpha^2}) e^{i\sqrt{\lambda-\alpha^2}x} + c_2 (\alpha - i\sqrt{\lambda-\alpha^2}) e^{-i\sqrt{\lambda-\alpha^2}x}]$$

$$\phi'(0) = 0 \Rightarrow c_1 (\alpha + i\sqrt{\lambda-\alpha^2}) = -c_2 (\alpha - i\sqrt{\lambda-\alpha^2})$$

$$\phi'(\pi) = 0 \Rightarrow c_1 (\alpha + i\sqrt{\lambda-\alpha^2}) e^{i\pi\sqrt{\lambda-\alpha^2}} = -c_2 (\alpha - i\sqrt{\lambda-\alpha^2}) e^{-i\pi\sqrt{\lambda-\alpha^2}}$$

"0 mode" 2nd eqn by the 1st to get:

$$e^{i\pi\sqrt{\lambda-\alpha^2}} = e^{-i\pi\sqrt{\lambda-\alpha^2}}$$

$$e^{2i\pi\sqrt{\lambda-\alpha^2}} = 1 \Rightarrow 2\pi\sqrt{\lambda-\alpha^2} = 2\pi n$$

This determines the eigenvalues $\lambda_n = n^2 + \alpha^2$ $n = 0, 1, 2, \dots$

We must still impose $c_1 (\alpha + i\sqrt{\lambda-\alpha^2}) = -c_2 (\alpha - i\sqrt{\lambda-\alpha^2})$

We can write $\alpha + i\sqrt{\lambda-\alpha^2} = \sqrt{\alpha^2 + n^2} e^{i\theta_n}$ $\theta_n = \tan^{-1}(n/\alpha)$

$$\text{Then } c_1 = \frac{1}{2i} e^{-i\theta_n} \quad c_2 = -\frac{1}{2i} e^{i\theta_n}$$

$n=0$ mode vanishes

In this way we determine the eigenfn's

$$\phi_0(x) = 1 \quad \lambda_0 = 0$$

$$\phi_n(x) = e^{\alpha x} \sin(nx - \theta_n) \quad \lambda_n = n^2 + \alpha^2 \quad n = 1, 2, \dots$$

Comments: i) $\sin(-nx - \theta_n) = -\sin(nx - \theta_n)$ so $n \rightarrow -n$ does not give a lin indep sol.

ii) Ex: Convince yourself that $\phi_1(x)$ has one internal zero, $\phi_2(x)$ has two internal zeros, etc.

The S-L thm guarantees that the set is complete, and we can expand initial dist. $u(x,0) = f(x)$ and $u(x,t)$ as

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} e^{\alpha x} \sin(nx - \theta_n)$$

$$\text{with coeffs } a_0 = \frac{1}{N_0^2} \int_0^{\pi} dx e^{-2\alpha x} f(x) \quad N_0^2 = \int_0^{\pi} dx e^{-2\alpha x}$$

$$a_n = \frac{1}{N_n^2} (\phi_n, f) = \frac{1}{N_n^2} \int_0^{\pi} dx e^{-2\alpha x} e^{\alpha x} \sin(nx - \theta_n) f(x)$$

$$\text{Ex: } N_n^2 = \int_0^{\pi} dx \sin^2(nx - \theta_n) = \frac{\pi}{2}$$

Long time limit: $u(x,t) \rightarrow a_0$ dependent only on $t \rightarrow \infty$

initial data. Suggests a cons. law. Previously we doubted this bc. we used wrong form of "energy". Correct this now by looking at $e^{-2\alpha x} \partial_t u = \partial_x [e^{-2\alpha x} \partial_x u]$

This suggests that the conserved energy for this prob. is $E = \int_0^\pi dx e^{-2\alpha x} u(x,t)$

Ex: Show that $\frac{dE}{dt} = 0$ if Neumann BC's are ^{satisfied}

Change of pace consider a different type of heat flow prob. How does temp. a few meters below surface of earth change due to seasonal variation at the surface.

Ref: Lm + Segel pp 150-151.

1. Assume surface temp described by simple periodic fn of t : $u(0,t) = A \cos \omega t$ where $T = \frac{2\pi}{\omega} \approx 1 \text{ yr.}$ (A more elaborate model of surf. temp might use Fourier ^{time series} $\sum_{n=0}^{\infty} A_n \cos n\omega t$)

2. Model earth as a medium with uniform diff. const k .

Assume ∞ depth, so we have the prob:

$$\text{PDE } \partial_t u = k \partial_x^2 u \quad \text{on } (0, \infty)$$

$$\text{BC } u(0,t) = A \cos \omega t$$

$$u(x,t) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad \leftarrow \begin{array}{l} \text{growth cond necessary} \\ \text{for a unique sol.} \end{array}$$

$$\text{I.C. } u(x,0) = f(x).$$

Comments a) model neglects effects of heat sources underground OK because they produce variations near surface on much longer time scales.

- b) We will also ignore the I.C., since the earth is a good conductor, and effect of initial temp will disappear on a short time scale $T' \ll T = 1 \text{ yr}$

Sol of Prob: An intelligent guess is that 'response' to the surface variation with freq ω is a subterranean variation of same frequency, so we assume

$U(x,t) = e^{i\omega t} \phi(x) + e^{-i\omega t} \phi^*(x)$. Then u is real and it must be, but we will work with $cr.$ $U(x,t) = e^{i\omega t} \phi(x)$ and then compute $u(x,t) = \text{Re } U(x,t)$ at the end. Since the PDE and BC are real, this will give sol. of orig. prob. (If you doubt this, you can check it post hoc.)

Subst $U(x,t) = e^{i\omega t}$ in PDE to get

$$i\omega \phi = k \partial_x^2 \phi \quad \text{since } e^{i\omega t} \text{ factors out,}$$

So we find the ODE $\phi'' = \frac{i\omega}{k} \phi$ with gen sol:

$$\phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x} \quad \text{with } \mu^2 = \frac{i\omega}{k} = e^{i\pi/2} \omega/k$$

$$\text{so } \mu = e^{i\pi/4} \sqrt{\frac{\omega}{k}} = (1+i)\sqrt{\frac{\omega}{2k}}$$

The C_1 term in sol will increase as $x \rightarrow \infty$ $C_1 e^{\frac{\sqrt{\omega}{2k}} x}$.

This is not physically sensible and violates the growth cond. So we set $C_1 = 0$, and take sol:

$$\phi(x) = C e^{-\sqrt{\frac{\omega}{2k}} x} e^{-i\sqrt{\frac{\omega}{2k}} x}$$

The real sol of our prob is then

$$U(x,t) = \text{Re} \left(e^{i\omega t} \phi(x) \right) = e^{-\sqrt{\frac{\omega}{2k}} x} \text{Re} \left(C e^{i(\omega t - \sqrt{\frac{\omega}{2k}} x)} \right)$$

Set $C = A$ to get

$$U(x,t) = A e^{-\sqrt{\frac{\omega}{2k}} x} \cos(\omega t - \sqrt{\frac{\omega}{2k}} x) \quad \text{satisfies BC at } x=0$$

Interp: a) sub. temp varies with same freq as at surface, but

b) amplitude of variation decreases with depth.

c) sub. temp is phase shifted by $\theta_x = \sqrt{\frac{\omega}{2k}} x$

$$\text{i.e. } u(x,t) = A e^{-\sqrt{\frac{\omega}{2k}} x} \cos(\omega t - \theta_x)$$

so at depth $\theta_x = \sqrt{\frac{\omega}{2k}} x_0 = \pi$, it is winter underground

when it is summer on the surface,

of seas. var

d) Realistic est. of $k \Rightarrow x_0 = 4.4$ meters, so amplitude u

attenuated by $e^{-\sqrt{\frac{\omega}{2k}} x_0} = e^{-\pi} \approx 0.04$, so seasonal

variation at depth x_0 is about 4% of surface variation.

This is the depth at which the ancient settlers of North America used to store their fruits and veggies.

Final word on scales: time scale $T = \frac{2\pi}{\omega}$ 1 yr.

spatial scale from $e^{-\sqrt{\frac{\omega}{2k}}x} = e^{-x/L}$, so $L = \sqrt{\frac{2k}{\omega}}$

Again: $T \sim L^2$.

Prepare for transition in 18.303 from heat flow probs in 1 space dim. to problems involving Laplacian ∇^2 in $d \geq 2$ space dim.
Ref Ch 7 of H. Start with heat Eq in finite region of plane

PDE: $\partial_t u(\vec{x}, t) = \nabla^2 u(\vec{x}, t) + Q(\vec{x}, t)$

$t \geq 0 \quad \vec{x} \in R$

BC: $u(\vec{x}, t) \Big|_{\vec{x} \in \partial R} = T(\vec{x}, t)$ when Dir.

or $\vec{n} \cdot \vec{\nabla} u(\vec{x}, t) \Big|_{\vec{x} \in \partial R} = \phi(\vec{x}, t)$ inhom. Neumann

or $\vec{n} \cdot \vec{\nabla} u(\vec{x}, t) \Big|_{\vec{x} \in \partial R} = -h(\vec{x}, t) [u(\vec{x}, t) - T(\vec{x}, t)]$ mixed BC

Note: $h(\vec{x}, t) = 0 \iff$ not outflow of heat if $u - T > 0$ cooling

I.C. $u(\vec{x}, 0) = f(\vec{x})$

Consider simplest case: $Q = 0$ no source, D on N $u|_{\partial R} = 0$ $\vec{n} \cdot \vec{\nabla} u|_{\partial R} = 0$

Solve by sep. of vars;

(1) Look for sols of form $u(\vec{x}, t) = T(t) \phi(\vec{x})$,

The PDE separates: $\frac{\dot{T}}{T} = \frac{\nabla^2 \phi}{\phi} = -\lambda$

Soln of T prob is $T = e^{-\lambda t}$ as usual,

(2) The ϕ prob is: $\nabla^2 \phi(\vec{x}) = -\lambda \phi(\vec{x}) \quad \vec{x} \in R$

BC $\phi(\vec{x})|_{\partial R} = 0 \quad D$

or $\vec{n} \cdot \vec{\nabla} \phi|_{\partial R} = 0 \quad N$

Eigen. prob for ∇^2 in 2 dim region R . This is classic 19th century mathematics. prob. Many features of 1-dim S-L prob are still valid. Let's summarize them: Sec 7.4 of H

(1) all eigenvalues of D prob are strictly +ve, $\lambda > 0$

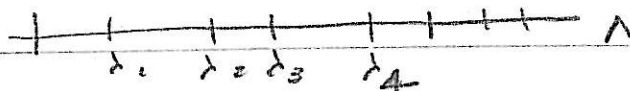
For N problem, we have $\lambda \geq 0$ and there is a zero

mode $\hat{\phi}_0(\vec{x}) = \frac{1}{\sqrt{A}}$

$A = \iint_R d^2x = \text{area of region}$

These statements can be proven by mult. and integ. args in 2-dim, (analogous to our work on S-L prob.)

- 2) so disc. spectrum of eigenvalues λ_n with at most a finite number in range $0 \leq \lambda \leq \Lambda$ for any $\Lambda > 0$



- 3) For each λ_n , there is a finite number of lin indep eigenfns: $\phi_{n,i}(\vec{r})$.

Orthogonality: if $\lambda_n \neq \lambda_m$ it is automatic that

$$(\phi_{n,i}, \phi_{m,j}) \equiv \iint_R d^2x \phi_{n,i} \phi_{m,j} = 0$$

For the same λ_n one can use Gram Schmidt to get an orthog set, so we can assume

$$(\phi_{n,i}, \phi_{m,j}) = N_{n,i}^{-2} \delta_{mn} \delta_{ij}$$

if $i = 1 \dots i_{\max}$ $i_{\max}$ may be large
we have "degenerate spectrum"

- 4) Set $\{\phi_{n,i}(\vec{r})\}$ is a complete orthog set, so any smooth $f(\vec{r})$ has expansion:

$$f(\vec{r}) = \sum_{n,i} a_{n,i} \phi_{n,i}(\vec{r}) \quad a_{n,i} = \frac{1}{N_{n,i}^2} (\phi_{n,i}, f)$$

$$(f, g) \equiv \iint_R d^2x f(\vec{r}) g(\vec{r})$$

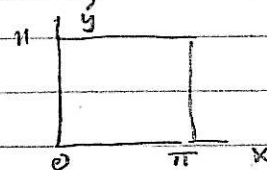
- 5) ~~Not~~ gen'l statements about zeros of $\phi_n(\vec{r})$ in contrast to 1-dim case.

We may prove some of these assertions later, first an

Example: Eigen prob on square

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(x,y) = -\lambda \phi(x,y)$$

$$\text{BC } \phi(x,y) \Big|_{\partial R} = 0$$



It should be clear that the eigenfns are products

$$\phi_{n,m}(x,y) = \sin nx \sin my \quad \lambda_{n,m} = n^2 + m^2$$

(Product structure holds because of symmetry of square, not true in general )

Lowest eigen is $\lambda = 2$ with unique $\phi_{1,1} = \sin x \sin y$

Next level is $\lambda = 5$ which is doubly degenerate

$$\phi_{1,2} = \sin x \sin 2y \quad \phi_{2,1} = \sin 2x \sin y$$

EB: Orthog $\iint \phi_{1,2} \phi_{2,1} = 0$