

Th Oct 15

Summary: 1. Green's fn for heat flow on $[0, \pi]$ for either

$$D \quad G_D(x, y, t) = \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

$$N \quad G_N(x, y, t) = \sum_{n=0}^{\infty} e^{-n^2 t} \cos nx \cos ny$$

allows you to solve

$$\text{IVP or } u(x, t) = \left(\int_0^{\pi} dy G(x, y, t) f(y) \right) \quad D \text{ or } N$$

2. $G(x, y, t)$ is itself the unique sol of
PDE $\partial_t G(x, y, t) = \partial_x^2 G(x, y, t)$ in x where

BC's D or N at $x=0, L$

$$\text{IC} \quad \lim_{t \rightarrow 0^+} G(x, y, t) = \delta(x-y)$$

$G(x, y, t)$ is a super smooth (well defined) fn for $t > 0$

One can prove rigorously that

$$\lim_{t \rightarrow 0^+} u(x, t) = \lim_{t \rightarrow 0^+} \int_0^{\pi} dy G(x, y, t) f(y) = f(x)$$

It is convenient to assume

$$\begin{aligned} &= \int_0^{\pi} dy \lim_{t \rightarrow 0^+} G(x, y, t) f(y) \\ &= \int_0^{\pi} dy \delta(x-y) f(y) = f(x) \end{aligned}$$

Result of this type of manip. is correct and can be justified using "theory of distributions"

3. We argued that $\delta(x-y)$ always appears as limit of smooth

$$\delta \text{ approximant: } \lim_{a \rightarrow 0} \Delta(x-y, a) = \delta(x-y)$$

One can prove rigorously that $\lim_{a \rightarrow 0} \int dy \Delta(x-y, a) f(y) = f(x)$

for any smooth "test fn" $f(y)$

Comment: The Green's fn for heat flow in $[0, \pi]$ with Neumann (insulator BC's) is $G_N(x, y, t) = \sum_{n=0}^{\infty} e^{-n^2 t} \cos nx \cos ny$.

Ex: Show (non-rigorously) that $\int_0^{\pi} dy G_N(x, y, t) = 1$

end Th

A non-rig. exercise to improve our understanding. On p 42 we used F. series method to find sol for Dirichlet heat flow for the initial rect. pulse

Note: $\int_0^{\pi} dy \text{ [pulse] } = 1$

Although [pulse] is not a smooth

fn. It should have the

S-approx. property as $\epsilon \rightarrow 0$

If in the limit $u(x, t)$ should describe future evolution

of a pt source. So we should expect $\lim_{\epsilon \rightarrow 0} u(x, t) = G(x, y, t)$

Our sol is:

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx \quad \text{with} \quad b_n = \frac{2}{n\pi\epsilon} \sin(n\epsilon) \sin ny$$

$$\lim_{\epsilon \rightarrow 0} u(x, t) = \sum_{n=1}^{\infty} \lim_{\epsilon \rightarrow 0} b_n e^{-n^2 t} \sin nx$$

To take $\lim_{\epsilon \rightarrow 0} b_n$ use l'Hopital's rule

$$\lim_{\epsilon \rightarrow 0} \frac{\sin(n\epsilon)}{\epsilon} = \lim_{\epsilon \rightarrow 0} \left(\frac{n \cos n\epsilon}{1} \right) = n$$

$$\text{Then } \lim_{\epsilon \rightarrow 0} b_n = \frac{2}{\pi} \sin ny$$

$$\text{Thus } u(x, t) \xrightarrow{\epsilon \rightarrow 0} \sum_{n=1}^{\infty} \frac{2}{\pi} \sin ny e^{-n^2 t} \sin nx = G_D(x, y, t)$$

Green's fn sol of heat flow with sources:

$$\partial_t u = \partial_x^2 u + Q(x, t)$$

Dirichlet BC's at $x=0, \pi$

$$\text{IC. } u(x, 0) = f(x)$$

Divide problem in two pieces (as in 18.03 for $y' + p y + q y = f(t)$)

$$y = y_p + c_1 y_1 + c_2 y_2$$

Ex: $u(x,t) = u_p(x,t) + u_{hom}(x,t)$

i) u_{hom} satisfies $\begin{cases} \text{PDE with no source} \\ \text{BC's (trivial)} \\ \text{IC } u_{hom}(x,0) = f(x) \end{cases}$

ii) u_p satisfies $\begin{cases} \text{PDE with source} \\ \text{BC's (trivial)} \\ \text{IC trivial } u_p(x,0) = 0 \end{cases}$

We already know $u_{hom} = \int_0^\pi dy \, G(x,y,t) f(y)$

We will solve the problem for u_p by using the

eigenfns $\phi_n(x) = \sin nx$ $u_n(x,t) = e^{-\lambda_n t} \phi_n(x)$
 $\lambda_n = n^2$

We will use the more gen'l notation, to emphasize that same method works for more gen'l heat flow probs.

Step 1: Expand both source + sol. in eigenfn series

$$Q(x,t) = \sum_{n=1}^{\infty} g_n(t) \phi_n(x) \quad g_n(t) = \frac{1}{N_n^2} (\phi_n, Q)$$

* $u_p(x,t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x)$

$$= \frac{2}{\pi} \int_0^\pi dy \, \phi_n(y) Q(y,t)$$

top form is more general

$$c_n(t) = \frac{1}{N_n^2} (\phi_n, u_p)$$

Step 2: Apply heat operator

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u = \sum_{n=1}^{\infty} \left[\dot{c}_n(t) \phi_n(x) - c_n(t) \frac{\partial^2}{\partial x^2} \phi_n(x) \right]$$

$$= \sum_{n=1}^{\infty} g_n(t) \phi_n(x)$$

Use orthog to obtain ODE for unknown coeffs c_n

$$\dot{c}_n + \lambda_n c_n = g_n(t)$$

18.03 1st order linear with integ. factor $p = e^{\lambda_n t}$

$$\frac{d}{dt} (e^{\lambda_n t} c_n) = e^{\lambda_n t} c_n$$

Unique sol satisfying the IC is

$$c_n(t) = e^{-\lambda_n t} \int_0^t dt' e^{\lambda_n t'} g_n(t')$$

$$c_n(0) = 0$$

Step 3: insert this sol for $c_n(t)$ in expansion * and exchange $\sum_n \leftrightarrow \int dt'$

$$u(x,t) = \int_0^t dt' \sum_n e^{-\lambda_n(t-t')} \phi_n(y) g_n(t')$$

$$\text{use } G \leftrightarrow \frac{1}{N_n^2} \int dy \phi_n(y) Q(y,t')$$

$$= \int_0^t dt' \int_0^\pi dy \sum_n e^{-\lambda_n(t-t')} \underbrace{\frac{\phi_n(x) \phi_n(y)}{N_n^2}}_{G(x,y,t-t')} Q(y,t')$$

$G(x,y,t-t')$ ← our new friend
Mr. Green

$$= \int_0^t dt' \int_0^\pi dy G(x,y,t-t') Q(y,t')$$

"Structure" of this Gm. for solution:

- i) universal, valid for any source
- ii) dynamics of 'system' is encoded in G , particular source appears only in last factor
- iii) sol is causal, i.e. value of $u(x,t)$ at obs. time t is determined by action of source at all earlier source times $t' \leq t$.

We have now developed the Gm. for sol for

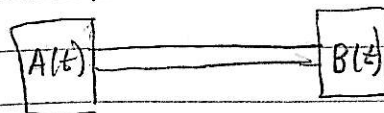
1) General IC, 2) General source There is another

interesting situation in heat flow, namely

Inhom. BC's eg $\partial_t u = \partial_x^2 u$ on $[0,1]$

$$\text{BC } u(0,t) = A(t)$$

$$u(1,t) = B(t)$$



Time dep. heat baths at end pts

This can also be handled quite elegantly using the same $G(x,y,t)$, but we will postpone our disc. of this situation.

(See Sec 8.4 of Text for (not so Green) treatment.)

Further disc of heat flow to emphasize generality of the ideas of sep. of variables $\rightarrow$ eigenfn. expansion $\rightarrow$ Green's fn.

Recall (p. 39) General heat eq. for non-uniform 2 dim region: $c(\vec{r}) \partial_t u(\vec{r}, t) = \nabla \cdot (K(\vec{r}) \nabla u)$ (no sources)

In 1 dim domain reduces to $c(x) \rho(x) \partial_t u(x, t) = \partial_x (K(x) \partial_x u)$

With some changes of notation, this can be made to look like a Sturm-Liouville situation on interval $a \leq x \leq b$.

$$\text{PDE} \quad w(x) \frac{\partial}{\partial t} u = \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x} \right)$$

$$\text{BC:} \quad u(a, t) = 0 \quad u(b, t) = 0 \quad \leftarrow \text{homog Dirichlet for simplicity}$$

$$\text{IC:} \quad u(x, 0) = f(x)$$

Program: i) Solve the prob. by sep of var + eigenfn. expansion
ii) put sol in Green's fn form.

Step 1: The sep sol of the PDE + BC's are

$$u_n(x, t) = e^{-\lambda_n t} \phi_n(x) \quad \text{where the } \{\phi_n(x)\} \text{ are the}$$

SL eigenfn. we studied in September.

$$\frac{d}{dx} \left(p(x) \frac{d\phi_n}{dx} \right) = -\lambda_n w(x) \phi_n(x)$$

$$\text{They satisfy orthog. } (\phi_n, \phi_m) \equiv \int_a^b dx w(x) \phi_n \phi_m = N_n^2 \delta_{nm}$$

Gen sol of PDE is superpos:

$$u(x, t) = \sum_{n=0}^{\infty} a_n u_n(x, t) = \sum_n a_n e^{-\lambda_n t} \phi_n(x)$$

Step 2: Find $\{a_n\}$ $n=0$ from initial data. Modern way, then process by using scalar prod. notation in the fn space $C[a, b]$

$$(f, g) = \int_a^b dx w(x) f(x) g(x)$$

$$\text{Then } (\phi_m, u) = \sum_n a_n e^{-\lambda_n t} (\phi_m, \phi_n) = N_m^2 a_m e^{-\lambda_m t}$$

This is valid for any $t > 0$. As $t \rightarrow 0^+$, we get $(\phi_n, f) = N_n^2 a_n$

Step 3: Find Green form of sol by subst a_n into $\sum_n$

$$u(x, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \frac{1}{N_n^2} (\phi_n, f)$$

$$= \int_a^b dy w(y) \sum_{n=0}^{\infty} e^{-\lambda_n t} \frac{\phi_n(x) \phi_n(y)}{N_n^2} f(y)$$

$$= \int_a^b dy w(y) G(x, y, t) f(y)$$

The Green's fn is $G(x, y, t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \frac{\phi_n(x) \phi_n(y)}{N_n^2}$

The structure of the more general problem is the same as for simpler case of infinite bar and Fourier sine series

Universal Spectral form of G : It is expressed in terms of eigenfunc + eigenvalues of spatial S-L problem: The same spectral form occurs in all situations of heat flow, even in $d=2, 3$ dim where the eigenfunc satisfy $\vec{\nabla} \cdot (p(\vec{r}) \vec{\nabla} \phi_n) = -\lambda_n \omega(\vec{r}) \phi_n(\vec{r})$

The role of S-L theory is to supply the complete set of eigenfunc.

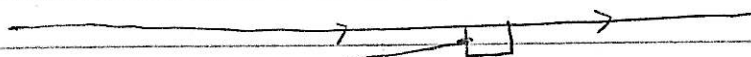
Study a new PDE $\partial_t u = \partial_x^2 u(x, t) - 2\alpha \partial_x u(x, t)$

Physical interp. is different depending on whether

$x \in \mathbb{R}$ so line or $x \in [a, b]$ finite interval

On $\mathbb{R}$ we will show that this describes a diffusion process in which particles simultaneously diffuse due to random collisions in a medium and also 'drift' with velocity $v_0 = 2\alpha$. Convection

Example River with current v_0



Nuclear power plant injects very hot water

$$\partial_t u = \partial_x^2 u(x, t) - 2\alpha \partial_x u + q(x, t)$$

To derive the convective picture, let's view the situation using new spatial coord. $y = x - v_0 t$ whose origin moves to right with velocity v_0 .

Define $\bar{u}(y, t) = u(x, t) = u(y + v_0 t, t)$ and use chain rule to transform to new variable:

$$\frac{\partial}{\partial x} u(x, t) = \frac{\partial y}{\partial x} \frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial y}$$

$$\text{Similarly } \partial_x^2 u = \frac{\partial y}{\partial x} \frac{\partial}{\partial y} \left(\frac{\partial \bar{u}}{\partial y} \right) = \partial_y^2 \bar{u}$$

$$\begin{aligned} \left. \frac{\partial}{\partial t} u(x, t) \right|_{x \text{ fixed}} &= \frac{\partial}{\partial t} \bar{u}(y(x, t), t) = \frac{\partial}{\partial t} \bar{u}(y, t) + \frac{\partial \bar{u}}{\partial y} \frac{\partial y}{\partial t} \\ &= \partial_t \bar{u}(y, t) - v_0 \frac{\partial \bar{u}}{\partial y} \end{aligned}$$

Subst these relations in pde