

$\Rightarrow$ lhs = 0 for any $t_0 > 0 \Rightarrow v(x, t_0) \equiv 0$

Then the sol. of original problem is unique:

Comments: a) very simple pf with a powerful conclusion

b) all elements of hypothesis were used.

c) another great "mult. + integ." argument.

d) Can be generalized to i) non-uniform heat flow, ii) sources
iii) BC: Newton's Law of Cooling

Important new direction in our course:

Ref. 0.9

GREEN's fn sol. for 'homog Dirichlet heat flow'. This method does not really permit us to solve concrete heat flow probs any better than sep. of vars, but it does give a lot of insight. (The method is also universal, but we won't emphasize this now.)

Heat flow on $[0, \pi]$ with $u(0, t) = u(L, t) = 0$

Previous sol: $u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} dy \sin ny f(y)$$

Insert formula for b_n in sum, exchange $\sum_n \leftrightarrow \int dy$

$$u(x, t) = \int_0^{\pi} dy f(y) \underbrace{\frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny}_{\equiv G(x, y, t)}$$

The Green's fn is defined by the series, which is rapidly converg.

if $t > 0$. So $G(x, y, t)$ is a well defined fn. (It is actually another special fn. related to the Jacobi θ -fn $\Theta(x-y, t)$)

Let's view G as known, then we can write sol. of heat flow prob as

$$u(x, t) = \int_0^{\pi} dy G(x, y, t) f(y).$$

Interp.: For any initial data $f(y)$, the future temp. distrib. $u(x, t)$

can be written as an integral with universal $G(x, y, t)$

Structure is similar to our Green's fn sol. of $Ly(x) = P(x)$

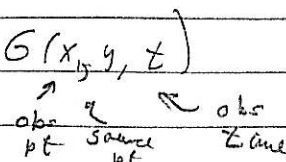
At first we used homog. soln $y_1(x), y_2(x)$ to write

$$G(t, s) = \frac{y_2(t)y_1(s) - y_1(t)y_2(s)}{v(y_1(s), y_2(s))}, \text{ but we}$$

then showed that $G(t, s)$ is unique sol. of IVP: $L_x G(x, s) = 0$

$$\text{with I.C.'s } G(t, t) = 0 \quad \partial_t G(t, s) \Big|_{s=t} = 1$$

⚡ The IVP no longer refers to $y_1(x), y_2(x)$



Do the same for heat flow: Start with eigenfn expansion

$$G(x, y, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

and derive an IVP for which G is unique sol.

1) Apply "heat operator" to G

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) G(x, y, t) = \frac{2}{\pi} \sum_n \underbrace{(\partial_t - \partial_x^2) e^{-n^2 t} \sin nx \sin ny}_{=0} = 0$$

Since $u_n(x, t) = e^{-n^2 t} \sin nx$

So $G(x, y, t)$ satisfies argument u a separable sol of PDE.

heat eq in x, t (or in y, t)

2) What IC does $G(x, y, t)$ satisfy

Take $t \rightarrow 0^+$ limit of sol $u(x, t) = \int_0^{\pi} dy G(x, y, t) f(y)$

We know from eigenfn. expansion that

$$u(x, t) \xrightarrow{t \rightarrow 0^+} f(x) = \lim_{t \rightarrow 0^+} \int_0^{\pi} dy G(x, y, t) f(y)$$

$$= \int_0^{\pi} dy \left(\lim_{t \rightarrow 0^+} G(x, y, t) \right) f(y) \quad \text{if } \lim_{t \rightarrow 0^+} \int dy = \int dy \lim_{t \rightarrow 0^+}$$

We will discuss this, but assume now that it is justified. Then

$$f(x) = \int_0^{\pi} dy G(x, y, 0) f(y) \quad \text{true for all choices of initial data } f(x)$$

So what must be the case is:

$$\lim_{t \rightarrow 0^+} G(x, y, t) = \delta(x - y) \quad \text{Dirac } \delta \text{ fn.}$$

Summarize: $G(x, y, t)$ is unique sol of the IVP

PDE: $\frac{\partial}{\partial t} G(x, y, t) = \frac{\partial^2}{\partial x^2} G(x, y, t)$

BC: $G(0, y, t) = 0 \quad G(\pi, y, t) = 0$

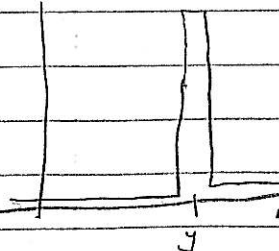
IC: $\lim_{t \rightarrow 0^+} G(x, y, t) = \delta(x - y) \quad *$

See Sec 9.2 of H. for an alternate discussion of G :

What is $\delta(x - y)$?

1. It is an object which (roughly speaking) "plots" as an infinitely narrow, infinitely narrow spike

An intuitive and tentative def.



Key property: For any smooth fn $f(x)$

$$\int dy \delta(x-y) f(y) = f(x) \quad **$$

In $**$ $\delta(x-y)$ appears as an I.C. in a heat flow problem.

Physical it describes a temp dist which is a "hot spot"

Temp is zero, everywhere on the bar, except at $x=y$.

Then we have a physical interp of: IVP for G . $G(x, y, t)$ is a sol of the heat eqn which describes the future evolution of an initial hot spot at y .

A more comprehensive discussion of $\delta(x)$:

- The δ fn almost always appears in applications as the limit $a \rightarrow 0$ of a family of smooth fns $\Delta(x-y, a)$ [In our case it appeared as the $\lim_{t \rightarrow 0} G(x, y, t)$]

One studies $F(x)$,

$$F(x, a) \equiv \int dy \Delta(x-y, a) f(y)$$

For $a \neq 0$, $F(x, a)$ is not the same as $f(x)$, but one can show rigorously that

$$\lim_{a \rightarrow 0} \int dy \Delta(x-y, a) f(y) = f(x)$$

If we formally allow

$$\lim_{a \rightarrow 0} \int dy \Delta(x-y, a) f(y) = \int dy \lim_{a \rightarrow 0} \Delta(x-y, a) f(y),$$

we would define the limit as $\lim_{a \rightarrow 0} \Delta(x-y, a) = \delta(x-y)$

and the key property $**$ would be true.

Moral of story: manipulations with δ -fns can be justified in integrals with smooth functions (called test fns.)

In most cases we are interested in $\delta(x-y)$ on ∞ line: $-\infty < x < \infty$

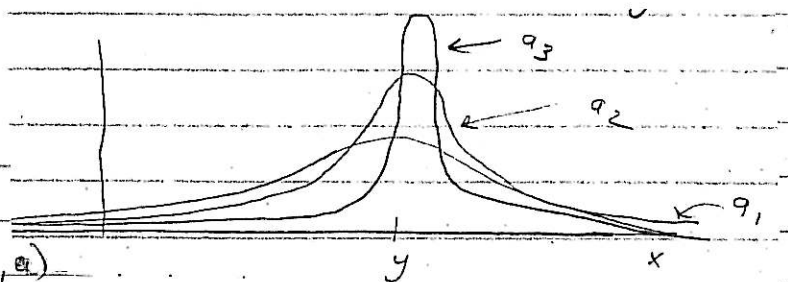
A common choice of $\Delta(x-y, a)$ is

$$\Delta(x-y, a) = \frac{a}{\pi} \frac{1}{(x-y)^2 + a^2} \xrightarrow{a \rightarrow 0} \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{if } x = y \end{cases}$$

Ex: $\int_{-\infty}^{\infty} dy \Delta(x-y, a) = 1$

for all a

Plot of
 $\Delta(x-y, a)$ for
 $a_1 > a_2 > a_3$



For every a , $\Delta(x-y, a)$
 is peaked at $x=y$. Peak
 becomes sharper and higher as $a \rightarrow 0$.

(Same area $\int \Delta \equiv 1$ for all a)

Show rigorously (by 18,100 methods) that

$$\lim_{a \rightarrow 0} \frac{a}{\pi} \int_{-\infty}^{\infty} dy \frac{f(y)}{(x-y)^2 + a^2} = f(x)$$

Proof: (from Prof. Jensen):

a. Assume f contin. + bounded on $\mathbb{R}$ $|f(x)| \leq M$

$$b. \frac{a}{\pi} \int_{-\infty}^{\infty} dy \frac{f(y)}{(x-y)^2 + a^2} = f(x) + \frac{a}{\pi} \int_{-\infty}^{\infty} dy \frac{f(y) - f(x)}{(x-y)^2 + a^2} \equiv f(x) + I(x, a)$$

Unit area property used.

To show: $I(x, a) \rightarrow 0$ as $a \rightarrow 0$

c. Change variable: $y = x + t$

$$I(x, a) = \frac{a}{\pi} \int_{-\infty}^{\infty} dt \frac{f(x+t) - f(x)}{t^2 + a^2}$$

d. Choose small t_0 and break interval into two pieces

$$I = \frac{a}{\pi} \int_{|t| > t_0} + \frac{a}{\pi} \int_{|t| < t_0}$$

outside peak inside peak at $t=0$

Use different estimate in each term:

e. Outside use $|f(x+t) - f(x)| < 2M$

$$|I_{out}| \leq \frac{2M}{\pi} \int_{|t| > t_0} \frac{a dt}{t^2 + a^2} = \frac{2M}{\pi} \int_{|u| > t_0/a} \frac{du}{u^2 + 1}$$

$$\leq \frac{2M}{\pi} \int_{|u| > t_0/a} \frac{du}{u^2} \leq \frac{4M}{\pi} \frac{a}{t_0}$$

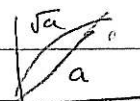
f. inside let $\tilde{M}(t_0) = \max_{|t| < t_0} |f(x+t) - f(x)|$
 $\rightarrow 0$ as $t_0 \rightarrow 0$

$$|I_{in}| \leq \tilde{M}(t_0) \frac{1}{\pi} \int_{|t| < t_0} \frac{adt}{t^2 + a^2} \leq \tilde{M}(t_0) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{adt}{t^2 + a^2} = \tilde{M}(t_0)$$

g. Put things together.

$$|I(a)| \leq \frac{4M}{\pi} \frac{a}{t_0} + \tilde{M}(t_0)$$

We still have the freedom to choose a relation between t_0 and a . Let $t_0 = c\sqrt{a}$, so that $t_0 \rightarrow 0$ as $a \rightarrow 0$, but more slowly.



$$\text{Then } |I(a)| \leq \frac{4M}{\pi} \sqrt{a} + \tilde{M}(\sqrt{a})$$

So $|I(a)|$ is bounded by a sum of 2 terms, and both $\rightarrow 0$ as $a \rightarrow 0$. Hence, with rigor, $\lim_{a \rightarrow 0} I(a) = 0$

So we have proved that $\Delta(x-y, a)$ is an approx. to $\delta(x-y)$.

Let's go back to Green sol of heat eq:

$$u(x,t) = \int_0^\pi dy G(x,y,t) f(y) \quad G(x,y,t) = \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

It is harder to prove rigorously that $\lim_{t \rightarrow 0^+} u(x,t) = f(x)$.

But our non-rigorous argument based on manip. of

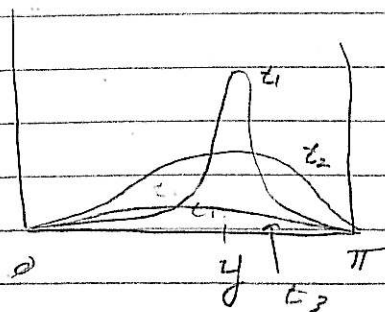
$$u(x,t) = \sum b_n e^{-n^2 t} \sin nx \text{ does give the correct conclusion}$$

To replace an earlier statement: physical interp is that

$G(x,y,t)$ describes diffusion of an initial pt. source of particles at $x=y$.

$$t_1 < t_2 < t_3$$

Note that $\int_0^\pi dy G(x,y,t)$ is not const., but $\rightarrow 0$ as t increases.



This is because there is no cons. law. for Dir. BC's