

Before launching into the math, ask physical question: what do we expect for $u(x,t)$ at large t .

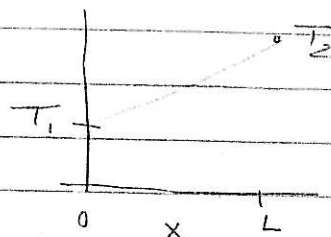
Heat will flow in and out of endpoints until: $u(x,t) \xrightarrow{t \rightarrow \infty} u_s(x)$
steady state (equil.) long time $\rightarrow$

Since $\partial_t u(x,t) \xrightarrow{t \rightarrow \infty} 0$, $u_s(x)$ must satisfy $\partial_x^2 u_s(x) = 0$

Gen sol $u_s(x) = ax + b$,

BC's are satisfied by unique choice

$$u_s(x) = T_1 + \frac{T_2 - T_1}{L} x$$



How to solve full problem: write $u(x,t) = u_s(x) + v(x,t)$.

Then v satisfies $\partial_t v = k \partial_x^2 v$ $\leftarrow$ same PDE

BC's $v(0,t) = 0$ $v(L,t) = 0$ $\leftarrow$ pure Dirichlet (homog)

IC $v(x,0) = f(x) - u_s(x)$ $\leftarrow$ modified IC. $\checkmark$

Essent. same prob. studied on pp 3-6 of notes.

Recall soln: $v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{(n^2\pi^2 k^2 t)}{L^2}}$

b_n are F. sine coeffs of initial data. $\checkmark$

Summarize physical picture:

use only fact that $v(x,t) \rightarrow 0$ as $t \rightarrow \infty$

1. At large t , sol of full prob $u(x,t) \rightarrow u_s(x)$
2. all memory of initial $f(x)$ is lost at large t
3. The $v(x,t)$ part does dep. on $f(x)$, but $v(x,t) \rightarrow 0$

Consider long time limit of heat flow with insulator BC's

At large t , we expect $u(x,t) = ax + b$ as before

BC's $\Rightarrow$ slope $a = 0$

Use conservation of total thermal energy to fix b .

$$E = c \int_0^L dx u(x,t)$$

$$\frac{\partial E}{\partial t} = c \frac{\partial}{\partial t} \int_0^L dx u(x,t) = ck \int_0^L dx \partial_x^2 u = ck \partial_x u \Big|_0^L = 0$$

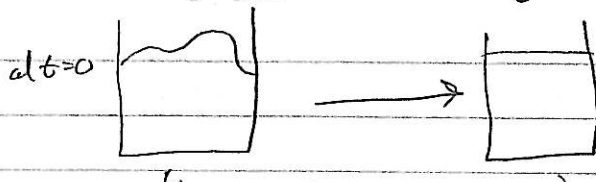
For insulating BC's (but not in general), total energy is indep of time.

Energy at $t = \infty$ = Energy at $t = 0$

$$\int_0^L dx b = \int_0^L dx u(x,t) \quad b = \frac{1}{L} \int_0^L dx u(x,t)$$

Summary: For heat flow with insulator BC's

$$u(x,t) \xrightarrow{t \rightarrow \infty} b = \frac{1}{L} \int_0^L dx f(x) \quad \text{mean value of initial temp dist.}$$



Tricky question: (for next PSet); what is long time limit of heat flow with non-homog flux BC's

$$\partial_x u(0,t) = -\frac{1}{k} \Phi_0 \quad \partial_x u(L,t) = -\frac{1}{k} \Phi_L$$

Hint use your head. The sol take 3-4 guidelines to write.

Return to Dirichlet heat flow (with ice water BC's) from pp 3-6:

Sol: from Sep. of vars. and eigenfn expansion:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2}{L^2} k t} \sin\left(\frac{n \pi x}{L}\right)$$

Comments

$$b_n = \frac{2}{L} \int_0^L dx \sin\left(\frac{n \pi x}{L}\right) f(x)$$

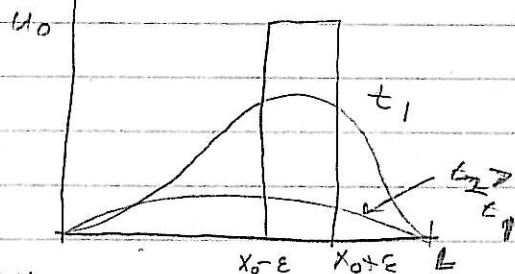
1. Convergence at $t=0$ determined by how smooth $f(x)$ is.

example: square spike of width 2ϵ centered at $x=x_0$, height u_0 .

Ex: Calculate

$$b_n = \frac{4u_0}{n\pi} \sin\left(\frac{n\pi\epsilon}{L}\right) \sin\left(\frac{n\pi x_0}{L}\right)$$

$$\sim 1/n \quad \text{as } n \rightarrow \infty$$



Slow convergence at $t=0$, but at any

small $t_1 > 0$ the effective F. coeff are $B_n = b_n e^{-\frac{n^2 \pi^2}{L^2} k t_1}$

So the series converges to supersmooth

temp, dist. No matter how 'jumpy' the initial data, the temp dist is supersmooth 1 ms later

Physical Principle: diffusion is a smoothing process:

2) For Dirichlet heat flow: $u(x,t) \rightarrow 0$ at large t , and dominate shape at large t is lowest eigenfn: $\phi_1(x) = \sin\left(\frac{\pi x}{L}\right)$

True because lowest mode has slowest falloff in t .

To bring this out more clearly, define a time scale $t_0 = \frac{L^2}{k\pi^2}$

$$\text{Then } T_n(t) = e^{-n^2 t/t_0}$$

For $t > 0$, the higher modes are damped by
 1) $1/n$ falloff of b_n , 2) $e^{-n^2 t/t_0}$ of $T_n(t)$ most important

Rough measure of importance of higher modes:

$$\frac{T_2(t)}{T_1(t)} = e^{-3t/t_0} = \begin{cases} .22 & \text{at } t = \frac{1}{2}t_0 \\ .05 & \text{at } t = t_0 \\ .0025 & \text{at } t = 2t_0 \end{cases} \quad u(x,t)|_{\partial R} = 0$$

It is a universal feature of bounded heat

flow in finite regions in any dim:

$u(\vec{r}, t) \rightarrow 0$ with shape of lowest eigenmode.

True for regions R of any shape and even for inhom. materials:

Note that eigenfun satisfy $\nabla^2 \phi_n(\vec{r}) = -\lambda_n \phi_n(\vec{r})$ $\phi_n(x)|_{\partial R} = 0$

Slightly diff. situation for homog. Neumann bc's (insulated)

Then $\lambda_0 = 0$ is an eigenvalue and

" $\phi_0 = L b_0 = \frac{1}{L} \int_0^L dx \phi(x)$ is fixed by energy cons.

Then $u(x,t) \xrightarrow{t \rightarrow \infty} b_0 + b_1 e^{-\lambda_1 k t} \phi_1(x)$

For uniform rod $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ See Sec 2.4 of H.

Another universal feature of heat flow (both D + N)

Time scale $t_0 = \frac{L^2}{k\pi^2} = \frac{1}{k\lambda_1}$ of diffusion process.

spatial scale $L = \text{length of rod}$.

Qualitative relation: time scale $\sim (\text{length scale})^2$

length scale $\sim (\text{time scale})^{1/2}$

This relation of the scales is quite universal, and we will encounter it in several ways in this course.

A bit more detailed example:

$$k_n = n^2 \pi^2 / L^2$$

def: spatial scale of n th mode $\sin(k_n x)$ is wavelength

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$$

Time scale of n th mode $T_n(t) = e^{-t/t_n}$ in $t_n = \frac{L^2}{k\pi^2 n^2}$.

AGAIN we see $t_n = \frac{1}{4k\pi^2} \lambda_n^2$ ← same relation

General remark about modeling physical processes:
 Important to identify the relevant scales in the problem.
 Much can be learned by combining this info with dimensional analysis.

Ex: Given heat eq $\frac{\partial}{\partial t} u = k \frac{\partial^2 u}{\partial x^2}$, we can
 change units of time by $t' = kt$,
 By chain rule $\frac{\partial u}{\partial t'} = \frac{\partial^2 u}{\partial x^2}$

Or can change units length by $x' = \frac{1}{\sqrt{k}} x$ and also
 derive $\frac{\partial}{\partial t} u = \frac{\partial^2 u}{\partial x'^2}$.

Conclusion: The diffusion const k can be eliminated by
 choice of units. So k is inessential for the mathematical
 analysis. No loss of gen. to take $k=1$.

A more theoretical issue: Uniqueness thm. for Heat Flow:

Let $u_1(x,t)$ and $u_2(x,t)$ be soln. of following prob.

- 1) Both satisfy uniform heat eq $\partial_t u = \partial_x^2 u$
- 2) " " same IC $u(x,0) = f(x)$ temp. $\downarrow$
- 3) " " same BC of form prescribed T : $u(0,t) = T_0(t)$ or
 prescribed flux $\partial_x u(0,t) = \phi_0(t)$, with same choice at $x=L$.
 (total of $2 \times 2 = 4$ distinct BC's)

Pf. Consider $V(x,t) \equiv u_1(x,t) - u_2(x,t)$. We want to show
 that $V(x,t) \equiv 0$. V satisfies

- 1) $\partial_t V = \partial_x^2 V$ same PDE by linearity
- 2) $V(x,0) = 0$ $\leftarrow$ trivial IC. $\checkmark$ homog BC's
- 3) either $V(0,t) = 0$ or $\partial_x V(0,t) = 0$, same at $x=L$

Step 1: Multiply PDE by V : $V \partial_t V = V \partial_x^2 V$
 $\frac{1}{2} \partial_t V^2 = V \partial_x^2 V$

Step 2: Integrate over spatial interval

$$\frac{1}{2} \int_0^L dx \partial_t V^2 = \int_0^L dx V \partial_x^2 V$$

$$= - \int_0^L dx \partial_x V \partial_x V + V \partial_x V \Big|_0^L$$

$\leftarrow$ last-rt term = 0 by homog. BC's

$$\text{Int. } \int_0^{t_0} dt : \quad \text{l.h.s.} = \frac{1}{2} \int_0^L dx V^2 \Big|_0^{t_0} = \frac{1}{2} \int_0^L dx V^2(x,t_0) \geq 0$$

$$\text{But r.h.s.} = - \int_0^{t_0} dt \int_0^L dx (\partial_x V)^2 \leq 0$$