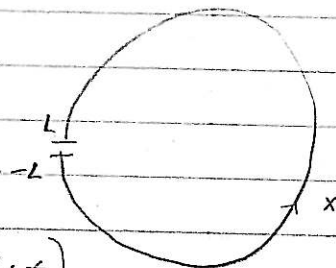


Last topic before PDE's:

Ref Haberman Sec 2.4.2
ch 3

Full Fourier series on $[-L, L]$: we will emphasize
cr. form of F. series.

Physical context of the prob: heat flow
on circular wire of length $2L$
 x is a coord along the circle.



Ex: PDE $\partial_t u(x, t) = k \partial_x^2 u(x, t)$

Assume separable soln $u = T(t) \phi(x)$

Derive eigenval. prob $\phi''(x) = -\lambda \phi(x)$ on $[-L, L]$

What are the BC's? The physical wire has no bdy,
but we introduce a "math. bdy" at $x = \pm L$, in order
to introduce the coord x .^{*} Clearly we must req. that
sols $\phi(x)$ are smooth across the "non-bdy"

$$\phi(L) = \phi(-L)$$

$$\phi'(L) = \phi'(-L)$$

← called periodic BC's

nat. of reg. S-L type, which will
have a modest effect on sols

* Note $x = \pm L$ denote same physical pt

Summary of Sec. 2.4.2 of H.

Eigenvalues of this prob are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 0, 1, 2, \dots$

The eigenfns are:

$$n = 0 \quad \phi_0(x) = 1$$

$$n > 0 \quad \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

$$\tilde{\phi}_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

← non-zero modes have 2-fold
degeneracy bc this is nat
a reg. S-L prob.

The modes are orthog. The expansion then is

$$* \quad f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$** \quad a_0 = \frac{1}{2L} \int_{-L}^L dx f(x)$$

$$a_n = \frac{1}{L} \int_{-L}^L dx \cos\left(\frac{n\pi x}{L}\right) f(x) \quad n > 0$$

$$b_n = \frac{1}{L} \int_{-L}^L dx \sin\left(\frac{n\pi x}{L}\right) f(x) \quad n > 0$$

Cr. Form of F. series: H. Sec 3.6

Motivated by Euler (de Moivre) formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Define:
$$c_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - ib_n) & n > 0 \\ \frac{1}{2}(a_{|n|} + ib_{|n|}) & n < 0 \end{cases}$$

Ex: By subst. + rearrangement, original series ~~is~~ can be written in equiv. cx. form

$$\checkmark \quad f(x) \equiv \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}}$$

Ex: By sust. + rearr. the formulas ~~is~~ for coeffs can be rewritten as

$$c_n = \frac{1}{2L} \int_{-L}^L dx e^{-i \frac{n\pi x}{L}} f(x)$$

The cx. F series is an expansion in the cx. eigenfns $\phi_n(x) = e^{i n \pi x / L}$

Ex: These fns are cx. orthog on $[-L, L]$

$$\int_{-L}^L dx \phi_m^*(x) \phi_n(x) = \int_{-L}^L dx e^{i(n-m)\pi x / L} = 2L \delta_{mn}$$

$$c_n = \frac{1}{2L} \int_{-L}^L dx \phi_n^*(x) f(x)$$

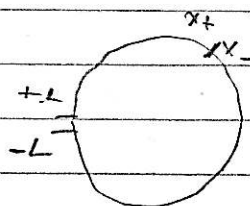
Summary + convergence: any piecewise smooth fn. $f(x)$, (both real or cx) has a conv. F series of the form $\checkmark$ with expain. coeffs given by $\checkmark$.

The truncated sum: $S_N(x) = \sum_{n=-N}^N c_n e^{i \frac{n\pi x}{L}}$ has the limit

$$S_N(x) \xrightarrow{N \rightarrow \infty} \begin{cases} f(x) & \text{pt. of continuity} \\ \frac{1}{2}(f(x_+) + f(x_-)) & \text{internal jump} \\ \frac{1}{2}(f(L/2) + f(-L/2)) & \text{at } x = \pm L \end{cases}$$

Ref Sec 3.2
of H

End conn. at end pt is same as inter. jup
becaus end pt is really just any
pt on circle



Example $f(x) = e^{ax}$ on $[-\pi, \pi]$



$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{-inx} e^{ax} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{(a-in)x}$$

$$= \frac{1}{2\pi} \left[\frac{1}{a-in} e^{(a-in)x} \right]_{-\pi}^{\pi} = \frac{(-1)^n}{2\pi(a-in)} (e^{a\pi} - e^{-a\pi})$$

$$= \frac{(-1)^n}{\pi} \frac{a+in}{a^2+n^2} \sinh a\pi$$

$$\frac{1}{2} \sinh y = \frac{1}{2} (e^y - e^{-y})$$

Then $f(x) = e^{ax}$ has F. series

$$\tilde{e^{ax}} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2} e^{inx}$$

$\tilde{f(x)}$ means the F. series which represents $f(x)$

What do we learn from this simple example?

- i) The c_n are cr., although $f(x)$ is real, but $c_{-n} = c_n^*$
We rearrange series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} \underbrace{(c_n e^{inx} + c_{-n}^* e^{-inx})}_{\text{real}}$$

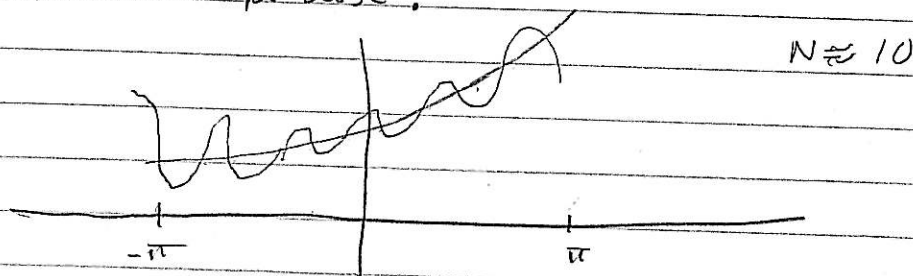
$\square$ and that series is real.

- ii) What happens at $x = \pm\pi$ use $\frac{\pm cn\pi}{2} = (-1)^n$

$$\tilde{e^{ax}} \Big|_{x=\pm\pi} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{+\infty} \frac{a+in}{a^2+n^2} \rightarrow \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2}$$

We see same limit at $x = \pm\pi$ in agreement with Conv. Thm.

- iii) rate of conv. F. coeffs $c_n \sim 1/n$ as $n \rightarrow \infty$. Another example of slow conv. (as in F. sine series of $\cos \frac{x}{2}$ in earlier handout (pp 8-9))
Worst conv. near end pt disc.



General disc. of convergence: rate of conv of
 $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ is determined by rate of decrease of
 c_n as $n \rightarrow \infty$. This is deter. by smoothness of $f(x)$.

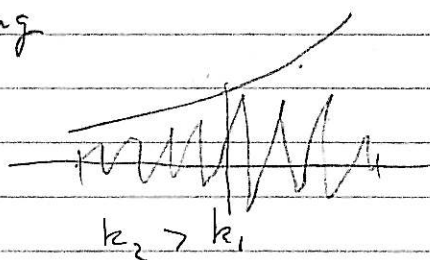
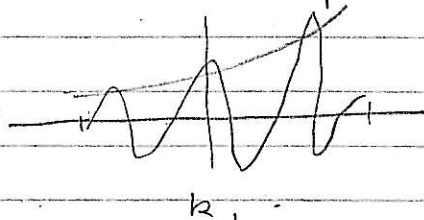
General situation governed by important Riemann-Lebesgue
Lemma: (very imp in rigorous work on F. series and
 eigenv. expansions)

Let $f(x)$ be a piecewise smooth function on $[a, b]$,

then the integrals $\int_a^b \frac{\cos kx}{e^{ikx}} f(x) dx \xrightarrow{\text{as } k \rightarrow \pm\infty} 0$

at least as fast as $1/k$.

Intuitive reason for vanishing



Integral has shorter wavelength $\lambda = \frac{2\pi}{k}$ as k increases,
 more and more cancellation betw. +ve and -ve areas.

Prob 11 on PSet 3; you will be asked to prove the R-L
 lemma. hint $e^{ikx} = \frac{1}{ik} \frac{d}{dx} e^{ikx}$.

Convergence rate improves with smoothness of $f(x)$,
 which is determined by how many derivs $\frac{d^m}{dx^m} f(x)$
 are contin. with no end-pt discs.

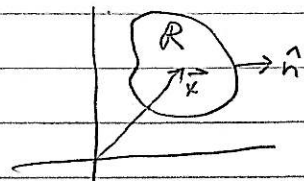
Finally PDE's, I hope that our foundation work on
 ODE's and eigenvalue probs will be worthwhile.

Heat Eq: $\partial_t u(\vec{x}, t) = k \nabla^2 u(\vec{x}, t)$ H. du

I. Derive PDE from General Balance Law

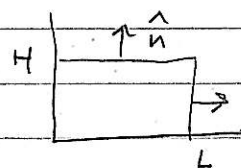
Let $a(\vec{x}, t)$ = density of "stuff" in place: $[a] = \frac{\text{stuff}}{\text{cm}^2}$

If applied to heat flow, the stuff will be thermal energy in the material.



Total amt. of stuff in region R at time t:

$$\int_R d^2x a(\vec{x}, t) \longrightarrow \int_0^L dx \int_0^H dy a(x, y, t)$$



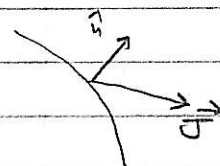
General Balance Law: time rate of change of

stuff in R is due to 1) flux of stuff through bdy of R ∂R
2) production rate due to sources inside R.

Flux is described by vect. $\vec{J}(x, t)$ in units $\frac{\text{stuff}}{\text{cm sec}}$.

such that $\Delta l \hat{n} \cdot \vec{J}$ is rate of flow through an element of arc on ∂R . $\hat{n}$ is unit outward normal to R.

Sources are described by a density $g(\vec{x}, t)$ in units of $\text{stuff}/\text{cm}^2 \text{ sec}$.



General Balance Law (integrated form).

$$\frac{\partial}{\partial t} \int_R d^2x a(\vec{x}, t) = - \int_{\partial R} \Delta l \hat{n} \cdot \vec{J} + \int_R d^2x g(\vec{x}, t)$$

A very general statement about production and loss of stuff

Use Gauss Law or divergence thm: $\int_{\partial R} \Delta l \hat{n} \cdot \vec{J} = \int_R d^2x \nabla \cdot \vec{J}$

Derivation of Gauss Law for the rectangle above.

$$\begin{aligned} \int_{\partial R} \Delta l \hat{n} \cdot \vec{J} &= \int_0^L dx [J_2(x, H, t) - J_2(x, 0, t)] + \int_0^H dy [J_1(L, y, t) - J_1(0, y, t)] \\ &= \int_0^H dy \int_0^L dx \partial_y J_2 + \int_0^L dx \int_0^H dy \partial_x J_1 \\ &= \int_R \nabla \cdot \vec{J} \end{aligned}$$

So we can rewrite gen. bal. law as a

$$\int_{\partial R} d^2x \left\{ \frac{\partial}{\partial t} a(\vec{x}, t) + \vec{\nabla} \cdot \vec{J} \right\} = \int_R d^2x g(\vec{x}, t)$$

This for any R , including any tiny subregion. So it must hold locally, so we derive Local Form of Gen. Bal. Law

$$\boxed{\frac{\partial}{\partial t} a(\vec{x}, t) + \vec{\nabla} \cdot \vec{J}(\vec{x}, t) = g(\vec{x}, t)} \quad (\text{also called eqn. of continuity})$$

A very gen. relation between density, flux and prod. rate of any quantity.

i) Our derivation was for $d=2$, but same form holds for $d=3, 1$.

ii) For $d=1$ flux is a 1-comp. vector $J(x, t)$, so we get

$$\frac{\partial}{\partial t} a + \frac{\partial}{\partial x} J = g(x, t)$$

Apply our result to heat flows in a material with

mass density $\rho(\vec{x})$ g/cm^3

specific heat $c(\vec{x})$ $\text{cal}/\text{gm}^\circ\text{K}$

thermal conductivity $K(\vec{x}) = \text{cal}/\text{cm}^\circ\text{K}$

$u(\vec{x}, t) = \text{temperature } ^\circ\text{K}$

Need "model" for thermal flux $\vec{J}$ which reflects simple physical features of heat flow:

a) no local heat flow if $\vec{\nabla} u = 0$

b) flow toward regions of lower temp.

Recall $\vec{\nabla} F$ is $\perp$ to level sets of $F(\vec{x})$,

points in direction of increasing F

Assume $\vec{J} = F(\vec{\nabla} u)$ where $F(s)$

should have properties: $F(0) = 0$

$F(s) < 0$ if $s > 0$; $F(s) > 0$ if $s < 0$

We don't know much about F (which depends on detailed molecular properties of the material).

But if $F(s)$ is smooth with $F(0) = 0$, we can approx. with linear form $F(s) \approx -\mu s$ where $\mu > 0$ depends on material.

In more detail we assume form $\vec{J}(\vec{x}, t) = -K(\vec{x}) \vec{\nabla} u(\vec{x}, t)$

Insert in gen. bal. law to get

$$\text{Heat Eq: } \boxed{c(\vec{x}) \rho(\vec{x}) \frac{\partial u}{\partial t} = \vec{\nabla} \cdot (K(\vec{x}) \vec{\nabla} u) + g(\vec{x}, t)}$$

Ex: check dimensions (units).

In most of our work, we will make simplifying assump. that material is homogeneous (uniform) $\Rightarrow c, \rho, k$ are constant

Heat eq $\rightarrow \boxed{\frac{\partial u}{\partial t} = k \nabla^2 u + g}$ $k = \frac{K}{c\rho}$ $[k] = \frac{\text{cm}^2}{\text{sec}}$ $\leftarrow$ Ex.

Note: spirit of our disc. was very empirical. We made the simplest assumptions which reflect the physical properties of the process we wanted to model.

Study: 1-dim heat flow without sources $\frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t)$

The sol. is unique (pt. later)
if we specify

I. initial temp $u(x,0) = f(x)$

II. BC's for prob in finite interval $[a,b]$

(For an ∞ line we req $u(\bar{x},t) \rightarrow 0$ as $\bar{x} \rightarrow \infty$)

3 Types of BC's: (needed for unique sol, and reflect physical cond of lab. exper.)

1. Prescribed Temp at endpoint

$$u(0,t) = u_B(t)$$



Heat reservoir with controlled temp

2. Prescribed Heat Flux

$$J(0,t) = -k \partial_x u(0,t) = \phi(t)$$



Flux barrier

3. Newton's Law of Cooling

$$J(0,t) = -k \partial_x u(0,t) = -H [u(0,t) - u_B(t)]$$

$H > 0$ to reflect fact that heat flux is outward (to left)
if $u - u_B > 0$.

a) One can have any of 3 BC's at $x=0$, and indep. 3 at $x=L$
Total of $3 \times 3 = 9$ distinct BC's

b) all related to inhom. versions of the D, N and mixed BC's

for S-L probs.

Ex: Newton at $x=L$ $J(L,t) = -k \partial_x u(L,t) = +H [u(L,t) - u_B(t)]$
 $\leftarrow H > 0$

Typical prob in 1-dim heat flow: on $[0,L]$

PDE $\partial_t u = k \partial_x^2 u$

BC's $u(0,t) = T_1$ $u(L,t) = T_2$ IC $u(x,0) = f(x)$