

Example $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$ $Y_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x \right)$ 21

For $\mu \neq n + 1/2$ the Bessel fns are transcendental. Defined as sols of ODE, and all properties can be deduced from ODE.

2) One way in which Bessel fns can appear in 18.303 is via Integral representations:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-in\theta} e^{ix \sin \theta} \quad n \text{ integer}$$

Ex: Show by diff. that Bessel ODE is satisfied

Ex: Show that $J_n(x) \sim x^n$ as $x \rightarrow 0$

New topic: Sturm-Liouville Probs. this is the setting in which ODE's occur in sep. of vars. for linear PDE's

General features 1) 2nd order linear ODE with bcy conds at 2 pts

2) ∞ # of orthog. eigenfns

3) Any (smooth) $f(x)$ has convergent eigenfn expansion

Systematic discussion in spirit of

Ref H. Ch 5

a) detailed summary of features of SL probs

b) prove some of key properties (but not all)

c) illustrate by examples

Define: a reg S-L prob on finite interval $[a, b]$ consists of

1. 2nd order lin ODE in form

$$L\phi = \frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) + q(x) \phi = -\lambda w(x) \phi$$

where $p(x)$, $q(x)$ and $w(x)$ are real contin. fns on $[a, b]$,

and $p(x)$, $w(x) > 0$

2. bc at $x=a$ $\phi(a) + k_a \phi'(a) = 0$

at $x=b$ $\phi(b) + k_b \phi'(b) = 0$

Properties:

1. The ODE + bc. prob. has solns only for an ∞ discrete set of real λ_n : $1 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$

$\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ λ is lower bound on the eigenvalue spectrum

2. For each eigenvalue λ_n , $\exists$ real eigenfn. $\phi_n(x)$

(unique up to $\phi_n(x) \rightarrow c \phi_n(x)$)

$\phi_n(x)$ has exactly $n-1$ zeros in (a, b) internal zeros

$\phi_{n+1}(x)$ " " " one zero between each pair of successive zeros of $\phi_n(x)$

3. Eigenfn are orthogonal on $[a, b]$ w.r.t weight fn $w(x)$

$$\int_a^b dx w(x) \phi_n(x) \phi_m(x) = \begin{cases} 0 & n \neq m \\ N_n^2 & n = m \end{cases} = N_n^2 \delta_{nm}$$

Kronecker δ

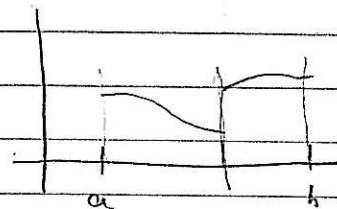
$N_n =$ norm of eigenfn

4. The set of eigenfn $\{\phi_n(x), n=0, 1, \dots\}$ is complete on $[a, b]$

$\Leftrightarrow$ Any well behaved $f(x)$ on $[a, b]$ is represented by convergent eigenfn expansion.

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

coeffs $a_n = \frac{1}{N_n^2} \int_a^b dx w(x) \phi_n(x) f(x)$



a piecewise smooth fn.

Convergence from truncated exp $S_N(x) = \sum_{n=0}^N a_n \phi_n(x)$

Then $S_N(x) \xrightarrow{N \rightarrow \infty} \begin{cases} f(x) & \text{if } f \text{ is contin at } x \\ \frac{1}{2}(f(x+) + f(x-)) & \text{avg of limits at a jump} \end{cases}$

$\frac{1}{2}(f(a) + f(b))$ at end pts

There is an elegant + powerful structure with many useful applics.

In many import. cases, one or more of the conds. assumed may not be true. e.g. $[a, b] \rightarrow [a, \infty]$ $\leftarrow$ semi- ∞ interval
 $w(x), p(x)$ can vanish at end pts.

In these cases we have a singular S-L problem. Mathematical pfr are more difficult, but there is a similar eigenfn. expan.

Comments on bc's vs IC's - some ODE might occur in a mechanics problem with $x \rightarrow t$

$$\frac{d}{dt} \left(p(t) \frac{dy}{dt} \right) + f(t) y(t) = h w(t) y(t)$$

but with IC's $y(t_0) = y_0, y'(t_0) = y_0'$

In this case $\exists$ a unique sol for any λ . For S-L probs, we have BC's at 2 end pts. This changes things a lot.

$\exists$ sols only for discrete eigenvalues $\{\lambda_n\}$

Start discussion whose purpose is partly to prove and partly to illuminate these features.

Derive Green's formula a basic tool to establish key properties

Let $u(x), v(x)$ be any two smooth fns on $[a, b]$

i. Consider $Lu = \partial_x(p \partial_x u) + q(x)u$

ii. Mult by v : $vLu = v \partial_x(p \partial_x u) + qvu$

$$= \partial_x(pv \partial_x u) - p \partial_x v \partial_x u + qvu$$

iii. $u \leftrightarrow v$ $uLv = \partial_x(pu \partial_x v) - p \partial_x u \partial_x v + qav$

iv. Subtract to get Lagrange identity:

$$v(x)Lu(x) - u(x)Lv(x) = \frac{d}{dx} [p(x)(v \partial_x u - u \partial_x v)]$$

v. Integrate and pick up end-pt contrib

$$\int_a^b dx [vLu - uLv] = p(x)(v \partial_x u - u \partial_x v) \Big|_a^b$$

Green's formula - pretty example of a "mult. and integ." argument common in linear ODE and PDE studies

vi) now assume $v(x)$ and $u(x)$ obey SL bdy cond. at $x=a, b$

$$\left(v \partial_x u - u \partial_x v \right) \Big|_a^b = \left(\partial_x v \partial_x u - \partial_x u \partial_x v \right) \Big|_a^b = 0$$

Conclusion

$$\int_a^b dx [vLu - uLv] = 0 \quad \text{for any pair of}$$

smooth $u(x), v(x)$ which satisfy B.C.'s but may or may not satisfy the ODE.

We say that SL operator is self-adjoint

Apply this to a pair of eigenfns: $u = \phi_n$ $L\phi_n = -\lambda_n w(x)\phi_n$

We get

$$v = \phi_m \quad L\phi_m = -\lambda_m w(x)\phi_m$$

$$0 = \int_a^b dx [\phi_m L\phi_n - \phi_n L\phi_m]$$

$$= - \int_a^b dx w(x) (\phi_m \lambda_n \phi_m - \phi_n \lambda_m \phi_m)$$

$$= (\lambda_m - \lambda_n) \int_a^b dx w(x) \phi_m \phi_n$$

If eigenvalues are distinct, eigenfns are orthogonal!

Related result: show that all λ_n are real

$$\text{use } L\phi = -\lambda w\phi$$

Mult. by ϕ^* and integrate

$$\int_a^b dx \phi^* [\partial_x (p \partial_x \phi) + q \phi] = -\lambda \int_a^b dx w \phi^* \phi$$

integrate by parts

$$\int_a^b dx [-p \partial_x \phi^* \partial_x \phi + q \phi^* \phi] + p \phi^* \partial_x \phi \Big|_a^b =$$

Restrict bc's to simplest types usually considered in 18.03

pure Dirichlet bc $\phi(a) = 0$

same at $x=b$

or pure Neumann bc $\phi'(a) = 0$

Total of $2 \times 2 = 4$ bc's in all 4 cases the end pt terms vanish (since ϕ^* obeys same bc)

Mult. everything by (-1) to get use $|\phi|^2 = \phi^* \phi$

$$\int_a^b dx p(x) |\partial_x \phi|^2 - \int_a^b dx q(x) |\phi|^2 = \lambda \int_a^b dx |\phi|^2 \quad v(x)?$$

1. The only way to satisfy this is with λ real

2. Supp. $q(x) \equiv 0$ (frequently true for PDE's in 18.303)

Then a) $\int_a^b dx w |\phi|^2 > 0$ and b) $\int_a^b dx p(x) |\partial_x \phi|^2 \geq 0$

vanishes only when $\phi = \text{const}$

Conclusion if $q(x) = 0$ then all eigenvalues $\lambda_n \geq 0$.

3. Further $\lambda = 0$ is possible only $\phi(x) = \text{const}$ is an eigenfn.

Not allowed with Dirichlet BC at either end,

But we have Neumann BC at both ends $\phi'(a) = \phi'(b) = 0$

then $\phi_0(x) = 1$ is an acceptable eigenfn with $\lambda_0 = 0$

4. Supp $q(x) < 0$ then $\lambda \Rightarrow$ all $\lambda_n > 0$

Physicists remember Schrodinger eq: $H\psi = -\psi'' + V(x)\psi = E\psi$

The corresp with our $\S L$ $L\phi = \phi'' + q\phi = -\lambda\phi$

is $q(x) \leftrightarrow -V(x)$ $\lambda \leftrightarrow E$

so $q(x) < 0 \Rightarrow V(x) > 0 \Rightarrow$ energy levels $E_n > 0$

Ex: Generalize 1, 2 to mixed BC's $\phi(a) + k_a \phi'(a) = 0$
same at b .

1' Show that λ is always real Hint Green

2' If $q(x) \equiv 0$, find constraints on k_a, k_b so that $\lambda \geq 0$

The most important fact about S-L problem is the completeness of the set $\{\phi_n(x)\}$ of eigenfn's.

This gives valid expansion (i.e.) $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$

It is neither hard to prove validity of such an expansion (although pf of Fourier series for piecewise smooth fns based on 18,100 ideas can be given in about 1 hr. of lecture time)

What we will do is to establish formula for the a_n , assuming that expansion is valid

(i) assume $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$ converges.

(ii) Mult by $W(x)\phi_m(x)$, and integrate $\int_a^b$, assume valid to exchange order of $\sum \leftrightarrow \int$.

$$\begin{aligned} \int_a^b dx W(x) \phi_m(x) f(x) &= \sum_{n=0}^{\infty} a_n \int_a^b dx W(x) \phi_m(x) \phi_n(x) \\ &= \sum_{n=0}^{\infty} a_n N_n^2 \delta_{nm} = a_m N_m^2 \end{aligned}$$

We have proved (non-rigorously) that

$$a_n = \frac{1}{N_n^2} \int_a^b dx W(x) \phi_n(x) f(x)$$

Normalization: since the eigen.

problem is linear $L\phi_n = -\lambda_n W\phi_n$, if $\phi_n(x)$ is

an eigenfn, so is $\tilde{\phi}_n = c_n \phi_n$ for any c_n and same λ_n .

Usually convenient to choose normalized eigenfn.

$$\hat{\phi}_n(x) = \frac{1}{N_n} \phi_n(x) \quad \text{so that} \quad \int_a^b dx W(x) \hat{\phi}_m(x) \hat{\phi}_n(x) = \delta_{nm}$$

Expansion formula simplifies to

$$\begin{aligned} f(x) &= \sum_n a_n \hat{\phi}_n(x) \\ a_n &= \int_a^b dx W(x) \hat{\phi}_n(x) f(x) \end{aligned}$$

Degression on analogy between mathematical structure of eigenvalue problem and vector spaces such as 3 dim Euc space R^3 .

Ref H. Appender to Sec 2.3 and App to Sec 5.5

Two abstract defs.: (then examples)

Def: a real vector space V is an ∞ set of objects $u, v, w, \dots$ on which two operations are defined.

1) addition if $u, v \in V$ then $u+v \in V$

2) scalar mult. if c is any real number and u any vector,

These operations define linear structures of V (then $cu \in V$ (u a vec))

One can also define additional structures on V ; such as a

Scalar product: a map $V \times V \rightarrow \mathbb{R}$ which obeys

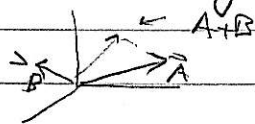
- i) $(u, v) = (v, u)$ symmetry
- ii) $(u+v, w) = (u, w) + (v, w)$ and $(cu, v) = c(u, v)$ linearity
- iii) if $(u, v) = 0 \quad \forall v$, then $u = 0$ non-degeneracy
- iv) $(u, u) \geq 0 = 0$ only if $u = 0$ positivity

These 4 properties are the abstr. def of a scalar product.

Vector spaces which realize the first def occur in many ways.

In each case we must specify the rules for 1) and 2) above.

Example 1: $\mathbb{R}^3$ realized geometrically in terms of vectors



or algebraically in terms of column vectors. $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

and defn $u+v = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$ $cu = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$

2. $C[a, b]$ vec. space of contin. fns on interval $[a, b]$

If $f(x)$ and $g(x)$ are contin fns, so are

$(f+g)(x) \equiv f(x) + g(x)$ pt. wise addition

$(cf)(x) \equiv cf(x)$ pt. wise multip.

These statements define the linear struct. of $C[a, b]$

(We know that the pointwise product $f(x)g(x)$ is also a contin. fn., but this is not part of the linear struct. Rather it is part of an additional structure of a Banach algebra

Scalar Products:

On $\mathbb{R}^3$ $(u, v) \equiv w_1 u^1 v^1 + w_2 u^2 v^2 + w_3 u^3 v^3$ $w_i > 0$

On $C[a, b]$ $(f, g)_w \equiv \int_a^b dx w(x) f(x) g(x)$ $w(x) > 0$

Tu Sept 28: Real vec. spaces with linear structure + scalar product.

Examples $\mathbb{R}^3 \rightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $C[a, b]$

Ex: Prove that $(f, g)_w$ on $C[a, b]$ has properties i-iv of a scalar product.

Note $(f, f)_w = \int_a^b dx w(x) (f(x))^2 \geq 0$

Can be viewed as $\text{length}^2 = \text{norm}^2$ of any vector.

Important practical idea - basis of V

A basis is a complete set of lin. indep. vecs. $e_1, e_2, \dots$, such that every $v \in V$ can be expr. as a unique superp. $v = \sum a_i e_i$

Most useful is an orthogonal basis: $(e_i, e_j) = N_i^2 \delta_{ij}$

Then $a_i = \frac{1}{N_i^2} (e_i, v)$

I. In $\mathbb{R}^3$ there are ∞^1 many ways to choose a basis and an orthog. basis.

1. non-orthog $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

2. std ON basis $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3. orthog $e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Every basis contains exactly 3 vecs.

Any set of 3 orthog vecs. is a basis.

The dimension of $\mathbb{R}^3$ is 3.

II What about $C[a, b]$? We know that the eigenfn's of any reg. S-L prob satisfy $L \phi_n = -\lambda_n w(x) \phi_n$ $w(x) > 0$

a) We have proven orthog. in the scalar product

$$(\phi_n, \phi_m)_w = N_n^2 \delta_{nm}$$

b. Mathematicians have proven eigenfun expansion

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x) \quad \text{with} \quad f_n = \frac{1}{N_n^2} (\phi_n, f)$$

So we say that set $\{\phi_n(x), n=0,1,2,\dots\}$ is an orthog basis of $C[a,b]$. Dimension of $C[a,b]$ is ∞

Summary: natural math. setting of SL eigenvalue prob in the ∞ -dim function space $C[a,b]$ with linear struct + scalar product above. The real miracle of SL prob is that the eigenfun $\{\phi_n(x)\}$ are an orthog basis in which any $f(x)$ has a unique convergent.

(Caution: our disc has been intuitive, far from rigorous. For a real mathematician the function space is not $C[a,b]$ but $L_2[a,b]$ - space of Lebesgue square integ. fns on $[a,b]$)

Function applic of basis in $\mathbb{R}^3$, then an analogue in $C[a,b]$

i. in usual ON basis $\hat{e}_i, i=1,2,3$, two vec's $\vec{A}, \vec{B}$

use vec. notation $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$\text{Expansions } \vec{A} = \sum_{i=1}^3 a_i \hat{e}_i \quad \vec{B} = \sum_{j=1}^3 b_j \hat{e}_j$$

$$\vec{A} \cdot \vec{B} = \left(\sum_i a_i \hat{e}_i \right) \cdot \left(\sum_j b_j \hat{e}_j \right) = \sum_{i=1}^3 a_i b_i$$

Usual formula for scalar prod. in terms of compns. of $\vec{A}, \vec{B}$

ii. Do exact analogue in $C[a,b]$ using ON basis

$$\{\hat{\phi}_n(x)\} \quad (\hat{\phi}_m, \hat{\phi}_n)_w = \delta_{mn}$$

$$\text{Expansions } u(x) = \sum_m u_m \hat{\phi}_m(x) \quad v(x) = \sum_n v_n \hat{\phi}_n(x)$$

$$(u, v)_w = \left(\sum_m u_m \hat{\phi}_m, \sum_n v_n \hat{\phi}_n \right) = \sum_{m,n} u_m v_n (\hat{\phi}_m, \hat{\phi}_n) \\ = \sum_n u_n v_n \quad 1.$$

$$\text{If } u=v, \quad (u, u)_w = \sum_{n=0}^{\infty} u_n^2 \quad 2.$$

Non-rigour, disc. since we assumed $\int_a^b \leftrightarrow \sum_{n,m}$, but
These useful formulas can be proven rigorously.

- 1. Plancherel formula
- 2. Parseval's formula

Comment: The results show the value of the $\mathbb{R}^3$ analogy

Two aspects of SL prob. not yet discussed.

- 1. Spectrum of eigenvalues. lower bound $\Lambda < \lambda_1 < \lambda_2 < \dots$
 ∞ # of λ_n , with $\lambda_n \rightarrow \infty$ no finite acc pt
- 2. $\phi_n(x)$ has exactly $n-1$ internal zeros.
 $\phi_{n+1}(x)$ has exactly one zero betw. each adj. pair of zeros of $\phi_n(x)$

Let's investigate these features:

A. Rigorous pf of lower bound from λ derived on p 24 of notes

For $\phi(x)$
$$\int_a^b dx p(x) (\phi')^2 - \int_a^b dx q(x) \phi^2 = \lambda \int_a^b dx w(x) \phi^2$$

(related to Rayleigh quotient see Sec 5.6 of H)

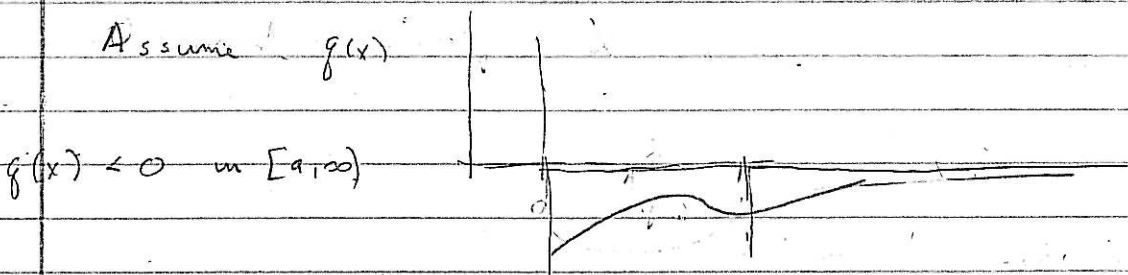
$$\lambda \int w \phi^2 \geq -q_{\max} \int \phi^2$$

$$\Rightarrow \lambda \geq \Lambda = -q_{\max} \frac{\int \phi^2}{\int w \phi^2}$$

B. Qualitative disc. of 1 and 2. To simplify take the case $p(x)=1$ and $w(x)=1$ and Dirichlet BC's $\phi(a)=0, \phi(b)=0$

Study eigenfn prob: $y'' + g(x)y = -\lambda y(x)$

Assume $g(x)$



Our bound on eigenvalue: $\lambda \geq \Lambda = -q_{\max} > 0$

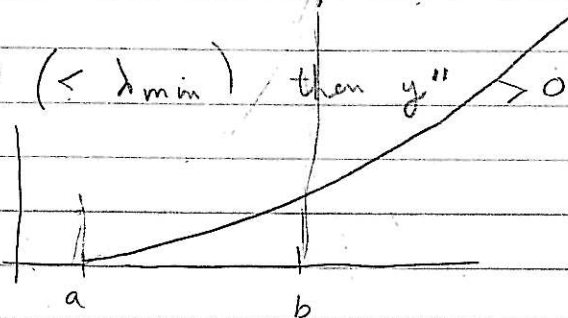
Rewrite ODE $y'' = -(\lambda + g(x))y$

- i) Recall from calculus - y'' is a measure of curvature of $y(x)$.
It tells us how fast the slope is changing
 $y'' > 0$ then $f(x)$ is concave upward
 $y'' < 0$ then $f(x)$ is concave downward
- ii) Consider the ODE $\square$ when λ is any real number (not necessarily an eigenvalue) and we

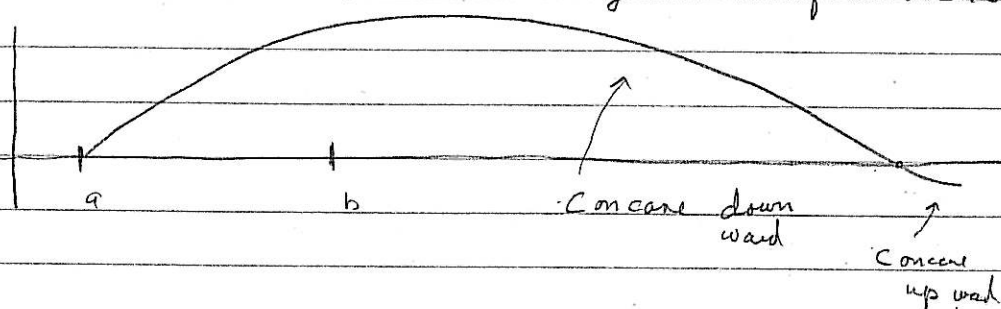
impose two IC's $y(a) = 0$ and $y'(a) = 1$

Consider the behavior of sol as λ changes. How can we adjust λ , so that BC $\phi(b) = 0$ so we have an eigenfn

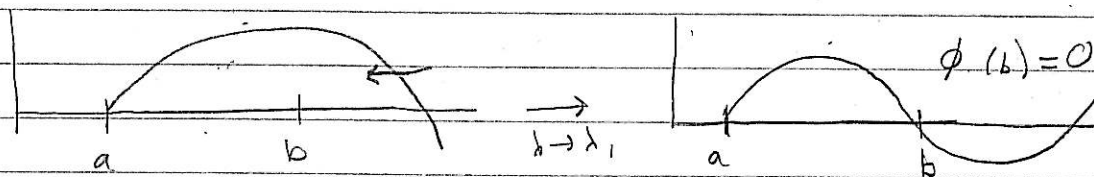
iii) if $\lambda = 0 (< \lambda_{\min})$ then $y'' > 0$



iv) as λ increases we reach pt where $y'' < 0$ for $x > c > b$

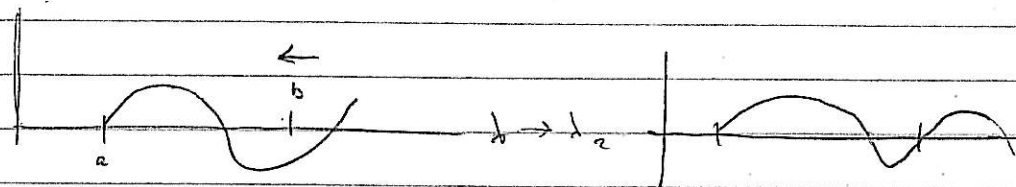


v) further increases in λ , first zero moves to left



As $\lambda \rightarrow \lambda_1$ we pick up first eigenfn $\phi_1(x)$ with another zero
no internal zeros visible at $x > b$

vi) increase λ again zeros move to left bc $|y''|$ increases

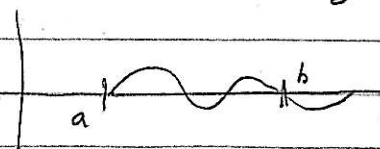


As $\lambda \rightarrow \lambda_2$ we pick up 2nd eigenfn $\phi_2(x)$ with 1 internal zero (and another zero for $x > b$)

vii) further increase brings second zero into $[a, b]$.

As $\lambda \rightarrow \lambda_3$ we pick up $\phi_3(x)$ with 2 internal zeros.

The process can be continued.



Note a cute fact which follows from orthog. (which we have proved.)

$$(\phi_1, \phi_2) = \int_a^b dx w(x) \phi_1(x) \phi_2(x) = 0$$



ϕ_1 with no zeros
 ϕ_2 must have at least one int. zero to be $\perp$ to ϕ_1 .

Ref: SL probs pp 133-137 of Simmons. He proves that $\lambda_n \xrightarrow{n \rightarrow \infty} \infty$ with no finite acc. pts.

Sturm Comparison Thm: consider sols of two ODE's

Simmons
p122

$$y'' + Q_2(x)y = 0$$

with $Q_2(x) > Q_1(x)$ in $[a, b]$

$$z'' + Q_1(x)z = 0$$

Then $y(x)$ vanishes at least once between any two successive zeros of $z(x)$

To apply this to SL probs take $Q_1(x) = \lambda_2 w(x) + q(x)$

$$Q_2(x) = \lambda_1 w(x) + q(x)$$

Proof let x_1, x_2 denote the zeros of $z(x)$.

No loss of generality to assume $z(x) \geq 0$ on $[x_1, x_2]$

* Then $z'(x_1) > 0$ and $z'(x_2) < 0$

Assume $y(x) > 0$ in (x_1, x_2) and

prove a contradiction.

Consider Wronskian-like quantity:

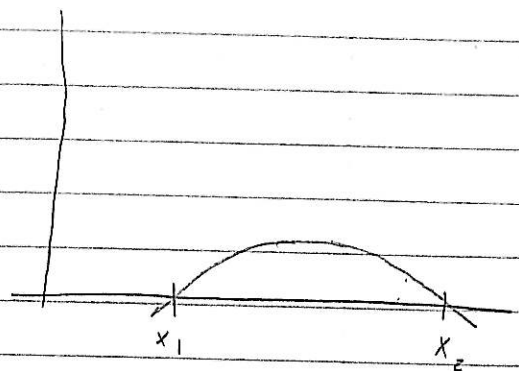
$$\bar{W}(x) = y(x)z'(x) - z(x)y'(x)$$

$$\frac{d\bar{W}}{dx} = yz'' - zy''$$

$$= y(-Q_1 z) - z(-Q_2 y) = (Q_2(x) - Q_1(x))y(x)z(x)$$

≥ 0 by our assumption in (x_1, x_2)

$$\bar{W}(x_2) - \bar{W}(x_1) = \int_{x_1}^{x_2} dx \frac{d\bar{W}}{dx} \geq 0$$



But from drawing $\bar{W}(x_1) = y(x_1)z'(x_1) \geq 0$

$$\bar{W}(x_2) = y(x_2)z'(x_2) \leq 0$$

Contradiction

Example of a non-trivial eigenv. prob of a type which can occur in PDE's:

Bessel ODE of order $\mu = 3/2$ on $[1, 2]$, let prev $x \rightarrow kx$

$$x^2 y'' + xy' - \frac{9}{4}y = -k^2 x^2 y$$

Dirichlet BC's

$$y(1) = 0 = y(2)$$

1) not quite in std SL form, but we can divide by x to get

$$\frac{d}{dx}(xy') - \frac{9}{4x}y = -k^2 xy \quad \text{so } w(x) = x \quad \text{and } h = k^2$$

This is a reg. SL prob on $[1, 2]$, so it should have a complete set of eigenfns with

$$\int_1^2 dx \, x \phi_n \phi_m = N_n^2 \delta_{nm}$$

The solns must be superpos. of those on p 21 of notes (with $x \rightarrow kx$)

$$J_{3/2}(kx) = \frac{1}{\sqrt{x}} \left(\frac{\sinh kx}{kx} - \cosh kx \right)$$

$$Y_{3/2}(kx) = \frac{1}{\sqrt{x}} \left(\frac{\cosh kx}{kx} + \sinh kx \right)$$

Now impose BC's on lin comb. $c_1 J_{3/2} + c_2 Y_{3/2}$

$$\text{at } x=1 \quad c_1 J_{3/2}(k) + c_2 Y_{3/2}(k) = 0$$

$$\text{at } x=2 \quad c_1 J_{3/2}(2k) + c_2 Y_{3/2}(2k) = 0$$

Rewrite as matrix eqn

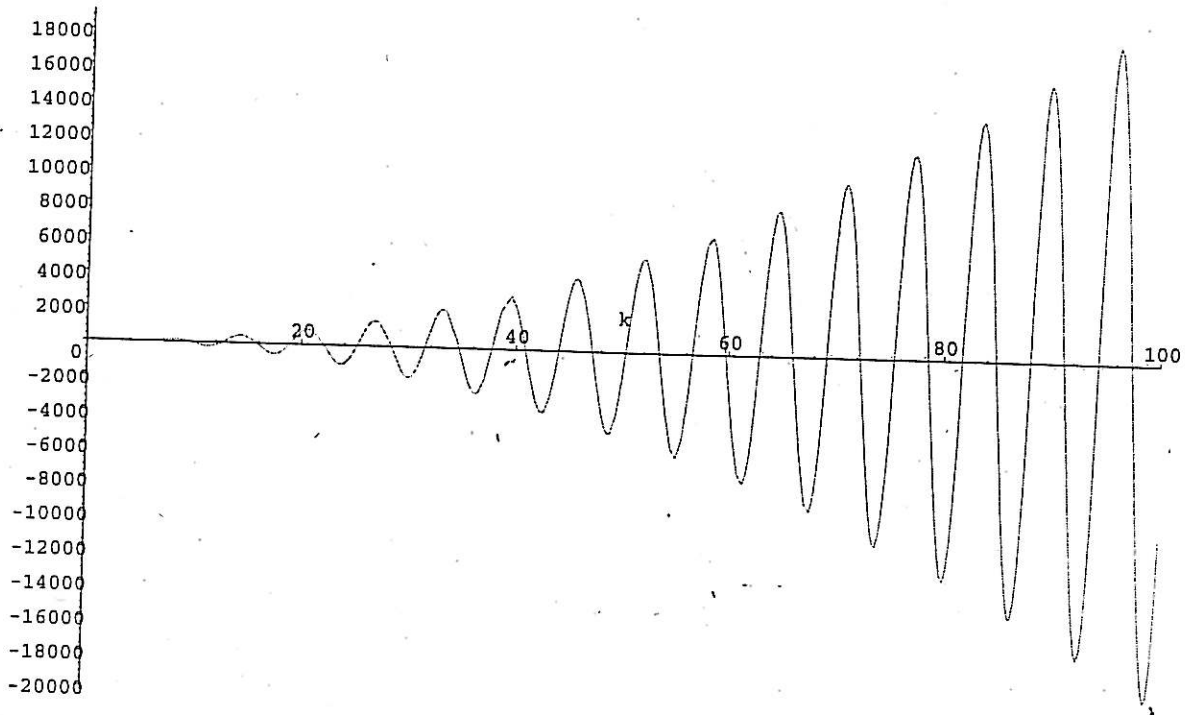
$$\begin{pmatrix} \frac{\sinh k}{k} - \cosh k & \frac{\cosh k}{k} + \sinh k \\ \frac{\sinh 2k}{2k} - \cosh 2k & \frac{\cosh 2k}{2k} + \sinh 2k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This prob has a sol iff. $\det M(k) = 0$. This is a transcendental eqn which can be explored numerically; using MAPLE.

However we can do one thing analytically:

$$\text{For } k \gg 1 \quad \det M(k) \approx -\cosh k \sinh 2k + \sinh k \cosh 2k = -\sinh k$$

So the large eigenvalues are $k_n = n\pi \quad n \gg 1$
 $\Delta k = \pi \leftarrow$ spacing of eigenvalues.



Maple plot of

$$k^3 \det M \equiv h(k) = (\sin 2k - 2k \cos 2k)(\cos k + k \sin k) - (\cos 2k + 2k \sin 2k) \cdot (\sin k - k \cos k)$$

vs k

The zeros are allowed eigenvalues of the Bessel eigenvalue problem discussed in class. Maple's `dsolve` command gives only one zero, at $k = 50.27543$.