

3. Eigenfn are orthogonal on $[a, b]$ w.r.t weight fn $w(x)$

$$\int_a^b dx w(x) \phi_n(x) \phi_m(x) = \begin{cases} 0 & n \neq m \\ N_n^2 & n = m \end{cases} = N_n^2 \delta_{nm}$$

Kronecker δ

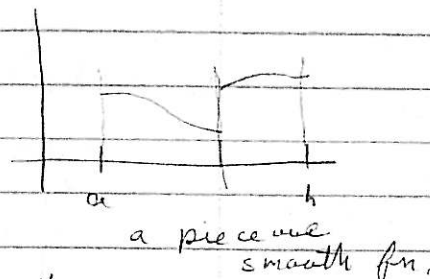
$N_n = \text{norm of eigenfn}$

4. The set of eigenfn $\{\phi_n(x), n=0, 1, \dots\}$ is complete on $[a, b]$

$\Leftrightarrow$ Any well behaved $f(x)$ on $[a, b]$ is represented by convergent eigenfn expansion.

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

coeffs $a_n = \frac{1}{N_n^2} \int_a^b dx w(x) \phi_n(x) f(x)$



Convergence from truncated exp $S_N(x) = \sum_{n=0}^N a_n \phi_n(x)$

Then
$$S_N(x) \xrightarrow{N \rightarrow \infty} \begin{cases} f(x) & \text{if } f \text{ is contin at } x \\ \frac{1}{2}(f(x_+) + f(x_-)) & \text{avg of limits at a jump} \\ \frac{1}{2}(f(a) + f(b)) & \text{at end pts} \end{cases}$$

This is an elegant + powerful structure with many useful appcs.

In many import. cases, one or more of the conds. assumed may not be true e.g. $[a, b] \rightarrow [a, \infty]$ $\leftarrow$ semi- ∞ interval
or $w(x), p(x)$ can vanish at end pts

In these cases we have a singular S-L problem. Mathematical pfr are more difficult, but there is a similar eigenfn. expan.

Comment on bc's vs IC's - some ODE might occur in a mechanics problem with $x \rightarrow t$

$$\frac{d}{dt} \left(p(t) \frac{dy}{dt} \right) + q(t) y(t) = h w(t) y(t)$$

but with IC's $y(t_0) = y_0 \quad y'(t_0) = y_0'$

In this case $\exists$ a unique sol for any λ . For S-L probs, we have BC's at 2 end pts. This changes things a lot.

$\exists$ sols only for discrete eigenvalues $\{\lambda_n\}$

Start discussion whose purpose is partly to prove and partly to illuminate these features.

Derive Green's formula a basic tool to establish key properties

Let $u(x), v(x)$ be any two smooth fns on $[a, b]$

i. Consider $Lu = \partial_x(p(x)\partial_x u) + q(x)u$

ii. Mult by v : $vLu = v\partial_x(p\partial_x u) + qvu$

$$= \partial_x(pv\partial_x u) - p\partial_x v\partial_x u + qvu$$

iii. $u \leftrightarrow v$ $uLv = \partial_x(pu\partial_x v) - p\partial_x u\partial_x v + quv$

iv. Subtract to get Lagrange identity:

$$v(x)Lu(x) - u(x)Lv(x) = \frac{d}{dx}[p(x)(v\partial_x u - u\partial_x v)]$$

v. Integrate and pick up end-pt contrib

$$\int_a^b dx [vLu - uLv] = p(x)(v\partial_x u - u\partial_x v) \Big|_a^b$$

Green's formula - pretty example of a "mult. and integ." argument common in linear ODE and PDE studies

vi) now assume $v(x)$ and $u(x)$ obey SL bdy cond at $x=a, b$

$$\left(v\partial_x u - u\partial_x v \right) \Big|_a^b = \left(\partial_x v\partial_x u - \partial_x u\partial_x v \right) \Big|_a^b = 0$$

Conclusion $\int_a^b dx [vLu - uLv] = 0$ for any pair of

smooth $u(x), v(x)$ which satisfy B.C.'s but may or may not satisfy the ODE.

We say that SL operator is self-adjoint

Apply this to a pair of eigenfns: $u = \phi_n$ $L\phi_n = -\lambda_n w(x)\phi_n$

We get

$$v = \phi_m \quad L\phi_m = -\lambda_m w(x)\phi_m$$

$$0 = \int_a^b dx [\phi_m L\phi_n - \phi_n L\phi_m]$$

$$= - \int_a^b dx w(x) (\phi_m \lambda_n \phi_m - \phi_n \lambda_m \phi_m)$$

$$= (\lambda_m - \lambda_n) \int_a^b dx w(x) \phi_m \phi_n$$

→ If eigen values are distinct, eigenfns are orthogonal!

Related result: Show that all λ_n are real

$$\text{use } L\phi = -\lambda w\phi$$

Mult. by ϕ^* and integrate

$$\int_a^b dx \phi^* [\partial_x (p \partial_x \phi) + q \phi] = -\lambda \int_a^b dx w \phi^* \phi$$

integrate by parts

$$\int_a^b dx [-p \partial_x \phi^* \partial_x \phi + q \phi^* \phi] + p \phi^* \partial_x \phi \Big|_a^b =$$

Restrict bc's to simplest types usually considered in 18.03

pure Dirichlet bc $\phi(a) = 0$

same at $x = b$

or pure Neumann bc $\phi'(a) = 0$

Total of $2 \times 2 = 4$ bc's in all 4 cases the end pt terms vanish (since ϕ^* always same bc)

Multiply everything by $(-)$ to get use $|\phi|^2 = \phi^* \phi$

$$\checkmark \int_a^b dx p(x) |\partial_x \phi|^2 - \int_a^b dx q(x) |\phi|^2 = \lambda \int_a^b dx |\phi|^2$$

1. The only way to satisfy this is with λ real

2. Supp. $q(x) \equiv 0$ (frequently true for PDE's in 18.303)

Then a) $\int_a^b dx w |\phi|^2 > 0$ and b) $\int_a^b dx p(x) |\partial_x \phi|^2 \geq 0$

vanishes only when $\phi = \text{const}$

Conclusion if $q(x) = 0$ then all eigenvalues $\lambda_n \geq 0$.

3. Further $\lambda = 0$ is possible only $\phi(x) = \text{const}$ is an eigenfn.

Not allowed with Dirichlet BC at either end,

But we have Neumann BC at both ends $\phi'(a) = \phi'(b) = 0$

then $\phi_0(x) = 1$ is an acceptable eigenfn with $\lambda_0 = 0$

4. Supp $q(x) < 0$ then $\checkmark \Rightarrow$ all $\lambda_n > 0$

Physicists remember Schrodinger eq: $H\psi = -\psi'' + V(x)\psi = E\psi$

The corresp with our S.L: $L\phi = \phi'' + q\phi = -\lambda\phi$

is $q(x) \leftrightarrow -V(x)$ $\lambda \leftrightarrow E$

so $q(x) < 0 \Rightarrow V(x) > 0 \Rightarrow$ energy levels $E_n > 0$

Ex: Generalize 1, 2 to mixed BC's $\phi(a) + k_a \phi'(a) = 0$
same at b .

1' Show that λ is always real Hint Green

2' If $q(x) \equiv 0$, find conditions on k_a, k_b so that $\lambda \geq 0$

The most important fact about S-L problem is the completeness of the set $\{\phi_n(x)\}$ of eigenfns.

This gives valid expansion (i.e.) $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$

It is rather hard to prove validity of such an expansion (although pf of Fourier series for piecewise smooth fns. based on 18,100 ideas can be given in about 1 hr of lecture time).

What we will do is to establish formula for the a_n , assuming that expan. is valid.

(i) assume $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$ converges.

(ii) Mult by $W(x)\phi_m(x)$, and integrate $\int_a^b$, assume valid to exchange order of $\sum \leftrightarrow \int$.

$$\begin{aligned} \int_a^b dx W(x) \phi_m(x) f(x) &= \sum_{n=0}^{\infty} a_n \int_a^b dx W(x) \phi_m(x) \phi_n(x) \\ &= \sum_{n=0}^{\infty} a_n N_n^2 \delta_{nm} = a_m N_m^2 \end{aligned}$$

We have proved (non-rigorously) that

$$a_n = \frac{1}{N_n^2} \int_a^b dx W(x) \phi_n(x) f(x)$$

Normalization: since the eigen.

problem is linear $L\phi_n = -\lambda_n W\phi_n$, if $\phi_n(x)$ is an eigenfn, so is $\tilde{\phi}_n = c_n \phi_n$ for any $\tilde{c}_n$ and same λ_n .

Usually convenient to choose normalized eigenfn.

$$\hat{\phi}_n(x) = \frac{1}{N_n} \phi_n(x) \quad \text{so that} \quad \int_a^b dx W(x) \hat{\phi}_m(x) \hat{\phi}_n(x) = \delta_{nm}$$

Expansion formula simplifies to

$$\begin{aligned} f(x) &= \sum_n a_n \hat{\phi}_n(x) \\ a_n &= \int_a^b dx W(x) \hat{\phi}_n(x) f(x) \end{aligned}$$

Digression on analogy between mathematical structure of eigenvalue problem and vector spaces such as 3 dim Euc space $\mathbb{R}^3$.

Ref H. Appendix to Sec 2.3 and App to Sec 5.5

Two abstract defs: (then examples)

Def: a real vector space V is an as set of objects $u, v, w, \dots$ on which two operations are defined.

1) addition: if $u, v \in V$ then $u+v \in V$

2) scalar mult. if c is any real number and u any vector,

These operations define linear structures of V (then $cu \in V$ (linear))

One can also define additional structures on V ; such as a

Scalar product: a map $V \times V \rightarrow \mathbb{R}$ which obeys

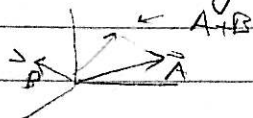
- i) $(u, v) = (v, u)$ symmetry
- ii) $(u+v, w) = (u, w) + (v, w)$ and $(cu, v) = c(u, v)$ linearity
- iii) if $(u, v) = 0 \quad \forall v$, then $u = 0$ non-degeneracy
- iv) $(u, u) \geq 0 = 0$ only if $u = 0$ positivity

These 4 properties are the abstr. def of a scalar product.

Vector spaces which realize the first def occur in many ways

In each case we must specify the rules for 1) and 2) above.

Example 1: $\mathbb{R}^3$ realized geometrically in terms of vectors



or algebraically in terms of column vectors. $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

and defn $u+v = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$ $cu = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix}$

2. $C[a, b]$ vec. space of contin. fns on interval $[a, b]$

If $f(x)$ and $g(x)$ are contin fns, so are

$(f+g)(x) \equiv f(x) + g(x)$ pt. wise addition

$(cf)(x) \equiv cf(x)$ pt. wise multip.

These statements define the linear struct. of $C[a, b]$

(We know that the pointwise product $f(x)g(x)$ is also a contin. fn., but this is not part of the linear struct. Rather it is part of an additional structure of a Banach algebra.