

Same is true for radial eqn in $d=3$ -space dim in sph. coords.

Sing pt $x=0 \leftrightarrow^{r=0}$ origin of coord system.

So behaviour of sols as $x \rightarrow 0$ is physically important.

Fortunately the sing is both Legendre + Bessel in a mild type and there is a simple method to find the behaviour of sols.

Def an 2nd order lin ODE is said to have a reg. sing. pt. at $x=x_0$ if $P(x)$ and $Q(x)$ are sing. there but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ are reg.

Frobenius method can be used to find behaviour of (at least one) sol. near $x=x_0$

First apply method to Bessel at $x=0$

$$\begin{cases} P(x) = 1/x \\ Q(x) = 1 - \mu^2/x^2 \end{cases}$$

Assume sol of form $y(x) = x^r z(x)$ where

$$z(x) = \sum_{n=0}^{\infty} b_n x^n \quad b_0 \neq 0 \quad \text{convergent for } |x| < x_0$$

Then $y' = r x^{r-1} z(x) + \mathcal{O}(x^r)$ as $x \rightarrow 0$
 $\mathcal{O}$ terms vanishing at least as fast as x^r as $x \rightarrow 0$

$$y'' = r(r-1)x^{r-2} z(x) + \mathcal{O}(x^{r-1})$$

Subst in ODE (best in non-std form)

$$(r(r-1) + r - \mu^2) x^r z(x) + \mathcal{O}(x^{r+1}) = 0$$

$\leftarrow$ dominant term as $x \rightarrow 0$

So a necessary cond for a sol of the eq. is $r^2 = \mu^2 \Rightarrow r = \pm \mu$

So we might guess that $\exists$ two lin indep sols of form

$$y_1(x) = x^{\mu} z_1(x) \quad \text{and} \quad y_2(x) = x^{-\mu} z_2(x)$$

Always true with $y_1(x) = J_{\mu}(x)$ $\leftarrow$ reg. Bessel fn in Sec 7.7-7.8 of H.
 Not always true

The actual situation is governed by thm. of Frobenius set things up:

1. Supp. that $y'' + P(x)y' + Q(x)y = 0$ has reg. sing. pt at $x=0$

$$x^2 y'' + p(x)y' + q(x)y = 0 \quad \begin{cases} p(x) = xP(x) \\ q(x) = x^2 Q(x) \end{cases}$$

2. Assume $\exists$ one sol with behaviour $y(x) = x^r z(x)$ as for

3. compute y' , y'' as before. Subst in ODE and find indicial eqn. $r(r-1) + r p_0 + q_0 = 0$

A necessary cond for a sol. of the type assumed is that r is a root of indicial eqn.

Suppose two real roots r_1, r_2 $r_1 > r_2$.

Thm: a) The ODE has one sol of the form

$$y_1(x) = x^{r_1} z_1(x) \quad z_1(x) \text{ has conv. pwr series with } z_1(0) \neq 0.$$

b) if $r_1 - r_2 \neq n$ $n=0, 1, 2, \dots$ then $\exists$ a 2nd sol of form $y_2(x) = x^{r_2} z_2(x)$ with $z_2(x)$ as above.

c) if $r_1 - r_2 = n$, the second sol has the form

$$y_2(x) = x^{r_2} z_2(x) + C \ln x y_1(x) \quad C \neq 0$$

Note Info on behavior of $y_2(x)$ is correctly given by

Red O formula: $y_2(x) = v(x) y_1(x)$

$$v(x) = \int \frac{x}{y_1^2(t)} dt = e^{-\int t p(t) dt}$$

Refs: Birkhoff-Rota Ch 9 Simmons

H. ch 7, especially for Bessel fns. Sec. 7.7.5

Note: There might be two cr roots $r = r_0 \pm i p_0 = r_{\pm}$

Then $\exists$ two cr. sols $Y_+(x) = x^{r_+} Z_+(x)$

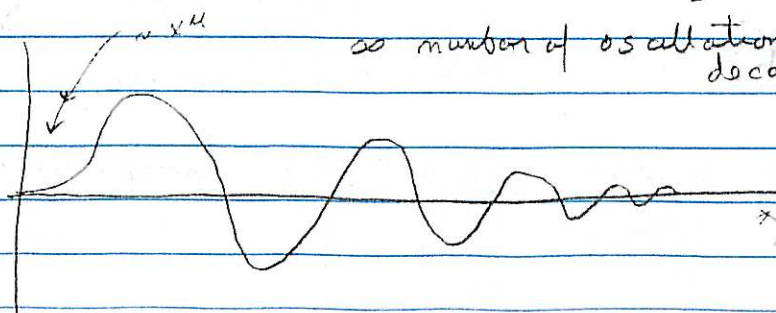
$$\text{and } Y_-(x) = Y_+(x)^*$$

Example: Regular Bessel fns. from Frobenius + pwr series method

$$\mu > 0 \quad J_{\mu}(x) = \left(\frac{x}{2}\right)^{\mu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\mu+n+1)} \left(\frac{x}{2}\right)^{2n} \quad \mu \text{ integer}$$

Ultrafast conv. series since $c_n \sim \frac{1}{2^n n! (n+\mu)!}$

Sketch $\sim x^{\mu}$ so number of oscillations within decaying envelope



Use Rof 0 to find behaviour of 2nd sol near $x=0$

Since $P(x) = \frac{1}{x} \quad \int^t P(s) ds \sim \ln t \quad \text{near } t=0$

Then $V(x) = \int_x^{x_0} \frac{dt}{J_\mu(t)^2 t} \quad e^{-\int^t P} \sim 1/t$

We take $x_0 < x_1 \leftarrow 1^{\text{st}} \text{ zero of } J_\mu$ and integrate down towards $x=0$

For $t < x_0$ we can represent

$$\frac{1}{J_\mu^2(t)} = \frac{1}{t^{2\mu}} \sum_{k=0}^{\infty} b_k t^k \quad \leftarrow \text{radius of convergence } t=x_1$$

(Can calculate coeffs b_n from series for $J_\mu(x)$, but no need for this)

If $\mu > 0, \neq n \quad V(x) \sim \int_x^{x_0} \frac{dt}{t^{2\mu+1}} \sum b_n t^n$

$$\sim \frac{V_0}{x^{2\mu}} + \frac{V_1}{x^{2\mu-1}} + \dots$$

So $y_2(x) = V(x) y_1(x) = x^{-A} Z_2(x)$ in agreement w. theorem.

If $\mu=0 \quad V(x) \sim \ln x + \text{regular}$

$y_2(x) = -\ln x J_0(x) + Z_2(x)$ as thm. requires

If $\mu=n \quad V(x) = \frac{V_0}{x^{2n}} + \frac{V_1}{x^{2n-1}} + \dots + \frac{V_{2n-1}}{x} + V_{2n} \ln x + \text{reg}$

$y_2(x) = \frac{1}{x^{2n}} Z_2(x) + V_{2n} \ln x J_n(x)$

Conclusion Rof 0 gives behaviour compatible with Frobenius in all cases.

It is useful and interesting to study behaviour of Bessel for $x \rightarrow \infty$
Use method of PSet 1.

Write $y(x) = x^s f(x)$ and choose s so that

$f(x)$ satisfies ODE of form $f''(x) + g(x)f = 0$

$$y' = s x^{s-1} f + x^s f'$$

$$y'' = s(s-1)x^{s-2} f + 2s x^{s-1} f' + x^s f''$$

Subst in original $x^2 y'' + x y' + (x^2 - \mu^2) y = 0$

$$x^5 [x^2 f'' + 2sx f' + s(s-1)f + (s^2 + x f')] + (x^2 - \mu^2)f = 0$$

$$= x^{s+2} \left[f'' + \frac{2s+1}{x} f' + \left(1 + \frac{s^2 - \mu^2}{x^2}\right) f \right] = 0$$

Set $s = -1/2$ to eliminate f' term. Result

$$y(x) = \frac{1}{\sqrt{x}} f(x) \quad \text{with} \quad f'' + \left(1 + \frac{1-4\mu^2}{4x^2}\right) f = 0$$

There is an exact eqn. $\nearrow$, which usually cannot be solved explicitly

But it is useful bc $1/x^2$ term is $\ll 1$ for large x .

So we should expect that

$$f(x) \xrightarrow{x \rightarrow \infty} h(x) + \tilde{r}(x) \quad \text{where}$$

$$h(x) \text{ satisfies } h'' + h = 0 \Rightarrow h(x) = c_1 \cos x + c_2 \sin x$$

$$\text{and } \tilde{r}(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\text{A more detailed analysis} \rightarrow f(x) = c \cos(x - \theta) + r(x)/x$$

There is what we need to conclude that $\nearrow r(x)$ bounded

$$y(x) \sim \frac{c}{\sqrt{x}} \cos(x - \theta) + r(x)/x^{3/2}$$

for x large

It is interesting that quite simple ideas have led to a good understanding of behaviour of Bessel fns.

(i) as $x \rightarrow 0$ from Froben. Thm ($\neq R=0$)

(ii) $1/\sqrt{x}$ envelope as $x \rightarrow \infty$

(iii) ∞ # of zeros spaced by π .

A more detailed study $\rightarrow$ reg. Bessel fn $J_\mu(x)$ (defined by series on p 18 (as p 323 of H.)) has asymp behaviour

$$J_\mu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \mu \frac{\pi}{2}\right) + \frac{r_1(x)}{x^{3/2}}$$

One makes a 2nd choice of second sol:

$$Y_\mu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \mu \frac{\pi}{2}\right) + \frac{r_2(x)}{x^{3/2}}$$

Comment: 1) look at exact ODE x for $\mu = 1/2$
 $f'' + f = 0$ is an exact eqn.

$$\text{Result } J_{1/2}(x) = \sqrt{\frac{2}{x}} \sin x \quad \text{and } Y_{1/2}(x) = \sqrt{\frac{2}{x}} \cos x$$

can be expressed in terms of elem. fns.

Same is true for $J_{n+1/2}(x)$ and $Y_{n+1/2}(x)$ $\leftarrow$ These are called spherical Bessel fns because they occur in 3D problems

Example $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$ $Y_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x \right)$

For $\mu \neq n + 1/2$ the Bessel fns are transcendental. Defined as sols of ODE, and all properties can be deduced from ODE.

2) One way in which Bessel fns can appear in 18.303 is via Integral representations:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-in\theta} e^{ix \sin \theta} \quad n \text{ integer}$$

Ex: Show by diff. that Bessel ODE is satisfied

EX: Show that $J_n(x) \sim x^n$ as $x \rightarrow 0$

New Topic: Sturm-Liouville Probs. this is the setting in which ODE's occur in sep. of vars. for linear PDE's

General features 1) 2nd order linear ODE with bcy conds at 2 pts

2) ∞ # of orthog. eigenfns

3) Any (smooth) $f(x)$ has convergent eigenfn expansion

Systematic discussion in spirit

Ref H. ch 5

a) detailed summary of features of SL probs

b) prove some of key properties (but not all)

c) illustrate by examples

Define: a reg S-L prob in finite interval $[a, b]$ consists of

1. 2nd order lin ODE in form

$$L\phi = \frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) + q(x) \phi = -\lambda w(x) \phi$$

where $p(x)$, $q(x)$ and $w(x)$ are real contin. fns on $[a, b]$.

and $p(x)$, $w(x) > 0$

2. bc at $x=a$ $\phi(a) + k_a \phi'(a) = 0$

at $x=b$ $\phi(b) + k_b \phi'(b) = 0$

Properties:

1. The ODE + bc prob. has unique sol for an ∞ discrete set of real λ_n : $1 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$

$\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ λ is lower bound on the eigenvalue spectrum

2. For each eigenvalue λ_n , $\exists$ real eigenfn. $\phi_n(x)$ (unique up to $\phi_n(x) \rightarrow c \phi_n(x)$)

$\phi_n(x)$ has exactly $n-1$ zeroes in (a, b) internal zeros

$\phi_{n+1}(x)$ " " " one zero between each pair of successive zeros of $\phi_n(x)$