

ii) Rewrite in matrix form: $\begin{pmatrix} \dot{y}_1 & \dot{y}_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} R(t) \\ 0 \end{pmatrix}$ 13

Algebraic eq.

$$M(t) \dot{v} = \begin{pmatrix} R(t) \\ 0 \end{pmatrix}$$

$$M(t) = \begin{pmatrix} \dot{y}_1 & \dot{y}_2 \\ y_1 & y_2 \end{pmatrix} \quad \det M = -W(y_1, y_2)$$

Sol $\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = M(t)^{-1} \begin{pmatrix} R(t) \\ 0 \end{pmatrix} = -\frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2 & -\dot{y}_2 \\ -y_1 & \dot{y}_1 \end{pmatrix} \begin{pmatrix} R(t) \\ 0 \end{pmatrix}$

So $v_1(t) = -\int \frac{y_2(t) R(t) dt}{W(t)} \quad v_2 = \int \frac{y_1(t) R(t) dt}{W(t)}$

iii) Put things together:

$$y_p(t) = y_1(t) \int \frac{-y_2(t') R(t') dt'}{W(t')} + y_2(t) \int \frac{y_1(t') R(t') dt'}{W(t')}$$

Note i) we should add integ. const's c_1, c_2 to each $\int$.

This changes $y_p(t) \rightarrow y_p(t) + c_1 y_1(t) + c_2 y_2(t)$

So it gives an equally valid $y_p(t)$.

ii) We could use prev. result $W(t) = C e^{-\int P(t) dt}$

but you need to find C from $W = y_1 \dot{y}_2 - y_2 \dot{y}_1$

Non Trivial example: $n=1$ Legendre with source

$$(1-x^2) y'' - 2x y' + 2y = f(x)$$

We already know $y_1(x) = x \quad y_2(x) = -1 + \frac{1}{2} x \ln \left(\frac{x+1}{x-1} \right)$

We need

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= x \left[\frac{1}{2} \ln \frac{x+1}{x-1} + \frac{1}{2} x \left(\frac{1}{x+1} - \frac{1}{x-1} \right) \right] - \left(-1 + \frac{1}{2} x \ln \frac{x+1}{x-1} \right) \cdot 1$$

$$= \frac{1}{2} x^2 \frac{-2}{x^2-1} + 1 = \frac{1}{1-x^2} \left(\frac{2}{2x} C e^{-\int P(x) dx} \right)$$

$$P(x) = \frac{2x}{1-x^2}$$

Use std. form
of ODE

also $R(x) = \frac{f(x)}{1-x^2}$

$$y'' + P y' + Q = R$$

MVP formula gives:

$$x \int \left(1 - \frac{1}{2} x \ln \frac{x+1}{x-1}\right) (1-x^2) \frac{f(x)}{1-x^2} dx$$

$$+ \left(-1 + \frac{1}{2} x \ln \frac{x+1}{x-1}\right) \int x (1-x^2) \frac{f(x)}{1-x^2} dx$$

Take $f(x) = 1$ to simplify integrals. Use MAPLE to get

$$y_p(x) = x \left(x - \frac{1}{2} (x-1) - \frac{1}{4} (x^2-1) \ln \frac{x+1}{x-1} \right) + \left(-1 + \frac{1}{2} x \ln \frac{x+1}{x-1} \right) \left(\frac{1}{2} x^2 \right)$$

$$= \frac{1}{2} x + \frac{1}{4} x \ln \frac{x+1}{x-1} = \frac{1}{2} y_1 + \frac{1}{2} y_2 + \frac{1}{2}$$

So $\hat{y}_p(x) = \frac{1}{2}$ is also a part. sol which could have been guessed.

What is important is that x is correct for any source.

Green's Function reinterpret. of MVP sol with new insight.

$$\ddot{y} + P(t) \dot{y} + Q(t) y = R(t)$$

✓ with $y_p(t) = V_1(t) y_1(t) + V_2(t) y_2(t)$

Use MVP results, but be careful to write integrals as

$$V_1(t) = \int_{t_0}^t ds \frac{y_2(s) R(s)}{W(y_1(s), y_2(s))} \quad V_2(t) = \int_{t_0}^t ds \frac{y_1(s) R(s)}{W(\dots)}$$

Subst in ✓ to get

$$y_p(t) = \int_{t_0}^t ds \frac{y_2(t) y_1(s) - y_1(t) y_2(s)}{y_1(s) \dot{y}_2(s) - y_2(s) \dot{y}_1(s)} R(s)$$

$$= \int_{t_0}^t ds \underbrace{G(t, s)}_{\downarrow} R(s)$$

Thus the Green's Fn. form of sol of inhom ODE. It is just a rewrite of MVP.

Interp. a) $G(t, s)$ encodes all info about diff. op

$$L y \equiv \left[\frac{d^2}{ds^2} + P(t) \frac{d}{dt} + Q(t) \right] y \quad \leftarrow \text{i.e. all info about } P(t), Q(t) \text{ which come from physical system}$$

b) Sol has universal form for any source

$G(t,s)$ is a ratio of dets.

$$G(t,s) = \frac{\begin{vmatrix} y_1(s) & y_1(t) \\ y_2(s) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(s) & \dot{y}_1(s) \\ y_2(s) & \dot{y}_2(s) \end{vmatrix}}$$

Does $G(t,s)$ depend on choice of lin indep sols y_1, y_2 ?

For example suppose we change to $\bar{y}_1, \bar{y}_2 = y_2 + cy_1$

How does G change?

$$\begin{aligned} \text{new Wronskian } W(\bar{y}_1, \bar{y}_2) &= \begin{vmatrix} y_1 & \dot{y}_1 \\ y_2 + cy_1 & \dot{y}_2 + c\dot{y}_1 \end{vmatrix} \\ &= W(y_1, y_2) + \begin{vmatrix} y_1 & \dot{y}_1 \\ cy_1 & c\dot{y}_1 \end{vmatrix} = W(y_1, y_2) \\ &= 0 \end{aligned}$$

So denom. of G is unchanged. Similar argument to show that numerator is unchanged.

We say that $G(t,s)$ is invariant.

Ex: Consider general change of basis $y_1(t), y_2(t)$ to

$$\bar{y}_1 = ay_1 + by_2 \quad \bar{y}_2 = cy_1 + dy_2$$

Show that $W(\bar{y}_1, \bar{y}_2) = (ad - bc) \cdot W(y_1, y_2)$

$$\text{num.}(\bar{y}_1, \bar{y}_2) = (ad - bc) \text{ num.}(y_1, y_2)$$

Important $G(t,s)$ is invariant under change of basis of homog sols y_1, y_2 . So it is intrinsically related to diff eq $L = \frac{d^2}{dt^2} + P(t) \frac{d}{dt} + Q(t)$

A pair of sols of $Ly = 0$ is needed to construct $G(t,s)$, but it does not matter which pair is used.

Some properties of $G(t,s)$

1. As a fn of t (with s fixed) $L_t G(t,s) = 0$ (but $L_s G(t,s) \neq 0$)
 $G(t,s)$ is a sol of homog ODE in t .
2. $G(t,s)|_{t=s} = 0$
3. $\partial_t G(t,s)|_{t=s} = 1$

These 3 conds define an IVP for $G(t,s)$ in which s plays role of time t_0 at which IC is specified

Theory of 2nd order lin ODE's guarantees that the sol. of this IVP is unique. Ref Birkhoff + Rota p 34

$\Rightarrow$ If unique $G(t,s)$ which satisfies props 1-3

Comments i) In 18.03 $\ddot{y} + p \dot{y} + q y = r(t)$ p, q constant

$G(t,s)$ is essentially the same as weight fn obtained by inverse Lap. Trf. of transfer fn.

Green's fn call $\leftrightarrow$ Duhamel's principle

ii) One $G(t,s)$ is not symmetric $G(t,s) \neq G(s,t)$

One can define a modified $G_{sym}(t,s)$ which is symmetric, but will not satisfy props 1-3. It is equally useful and can be found in some texts e.g. (Birkhoff-Rota p 98-99 ? p 76)

iii) Causality

$$y_p(t) = \int_{t_0}^t ds G(t,s) R(s)$$

Suppose t is a time variable, our notation implicitly assumes $t_0 < s < t$. Phys. interp. t is time at which we observe state of system, s is time at which source acts. So state of system at time t is affected by source at all earlier times.

Ex: what IC's are satisfied by sol $\checkmark \checkmark$?

Show that $y_p(t_0) = 0$ and $\dot{y}_p(t_0) = 0$ trivial IC's

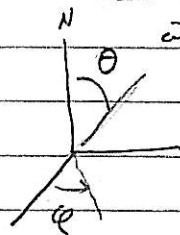
New topic: Behaviour of sols of 2nd order lin ODE near a regular singular pt.

Legendre eq in std form: $y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0$

The fns $P(x) = \frac{-2x}{1-x^2}$ $Q(x) = \frac{n(n+1)}{1-x^2}$ are singular at $x = \pm 1$

In phys. applics $x = \cos \theta \rightarrow \pm 1$ at N/S poles of sph. coord. syst.

So behaviour of sols at $x = \pm 1$ is important



Bessel's eq $[x^2 y'' + x y' + (x^2 - n^2) y = 0]$

in std form $P(x) = \frac{1}{x}$ and $Q(x) = 1 - \frac{n^2}{x^2}$ sing at $x=0$

Bessel in sep of vars. for 2d PDE's $\partial_z u = k \nabla^2 u$ $\nabla^2 = \partial_x^2 + \partial_y^2$ if we look for sep sols in polar coords r, θ .

$u(r, \theta) = G(t) f(r) h(\theta)$ $r \sim x$ of Bessel